Working with Exponents

Multiplying terms

When multiplying expression raised to powers, add exponents, like below:

$$x^a x^b = x^{a+b}.$$

Here are some examples:

1. $x^2 x^5 = x^7$
2. $y^3 y^9 = y^{12}$
3. $2^5 \cdot 2^6 = 2^{11}$
4. $2^x \cdot 2^3 = 2^{x+3}$
5. $x \cdot \sqrt{x} = x \cdot x^{1/2} = x^{3/2}$

Dividing terms

When dividing, subtract exponents, like below:

$$\frac{x^a}{x^b} = x^{a-b}.$$

Here are some examples:

1. $\dfrac{x^8}{x^5} = x^3$
2. $\dfrac{a^2}{a^6} = a^{-4}$ or $\dfrac{1}{a^4}$

Raising powers to powers

When raising a power to a power, multiply exponents, like below:

$$(x^a)^b = x^{ab}.$$

Here are some examples:

1. $\left(x^3\right)^5 = x^{15}$.
2. $\sqrt{x^5} = (x^5)^{1/2} = x^{5/2}$
3. $(2^x)^4 = 2^{4x}$

Minor technicality: We do need to be a little careful when square roots are involved. For instance $\sqrt{x^2}$ can be rewritten as $(x^2)^{1/2}$, which seems like it should equal x, but if x is negative, then that's not quite the case. The rule here is $\sqrt{x^2}$ equals x if x is positive and $-x$ if x is negative. Sometimes this is written as $\sqrt{x^2} = |x|$.

Breaking things up

A very common algebraic operation with powers is to break things up. The rules are

$$(xy)^n = x^n y^n \qquad\qquad \left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}.$$

Here are a few examples:

1. $(xy)^2 = x^2y^2$
2. $\left(\frac{x}{y}\right)^4 = \frac{x^4}{y^4}$
3. $(5x^3)^2 = 5^2 \cdot (x^3)^2 = 25x^6$
4. $\sqrt{5x} = \sqrt{5}\sqrt{x}$
5. $\sqrt[3]{\frac{x^3}{8}} = \frac{\sqrt[3]{x^3}}{\sqrt[3]{8}} = \frac{x}{2}$

Odds and ends

Here are a few further useful tricks:

1. Sometimes it's helpful to use the rules in reverse. For instance, we know $x \cdot x^4 = x^5$. So we could flip things and take x^5 and break it up into $x \cdot x^4$.
2. Sometimes we will want to move something into a root. For instance, $2\sqrt{x} = \sqrt{4y}$ or $x\sqrt{y} = \sqrt{x^2y}$ (for $x \geq 0$).

Exercises

1. Write the following as $x^{\square}$ where $\square$ is filled in with a single number.
   (a) x^3x^9
   (b) $\dfrac{x^{12}}{x^4}$
   (c) $(x^3)^5$
   (d) $\left(\dfrac{x^3}{x^5}\right)^2$
   (e) $x\sqrt[3]{x}$
2. Show that the following are true:
   (a) $(4x^2)^3 = 64x^6$
   (b) $\sqrt{9x^2} = 3x$ (for $x > 0$)
   (c) $\left(\dfrac{x^3}{9}\right)^2 = \dfrac{x^6}{81}$
3. What number fills in the box in $x^{27} = \left(x^3\right)^{\square}$?
4. What number fills in the box in $x^{27} = x^{\square}x^{25}$?
5. Rewrite $x^2\sqrt[3]{y}$ so that everything is under the radical.

Answers

1. (a) $x^3x^9 = x^{3+9} = x^{12}$

   (b) $\dfrac{x^{12}}{x^4} = x^{12-4} = x^8$

   (c) $(x^3)^5 = x^{3\cdot5} = x^{15}$

   (d) $\left(\dfrac{x^3}{x^5}\right)^2 = \left(x^{-2}\right)^2 = x^{-4}$

   (e) $x\sqrt[3]{x} = x \cdot x^{1/3} = x^{1+1/3} = x^{4/3}$ or $\sqrt[3]{x^4}$

2. (a) $(4x^2)^3 = 4^3(x^2)^3 = 64x^6$

   (b) $\sqrt{9x^2} = \sqrt{9}\sqrt{x^2} = 3x$ (for $x > 0$)

   (c) $\left(\dfrac{x^3}{9}\right)^2 = \dfrac{(x^3)^2}{9^2} = \dfrac{x^6}{81}$

3. 9 because $\left(x^3\right)^9 = x^{3\cdot9} = x^{27}$.

4. 2 because $x^2x^{25} = x^{2+25} = x^{27}$.

5. Rewrite x^2 as $\sqrt[3]{x^6}$ and then combine the radicals to get $\sqrt[3]{x^6y}$.