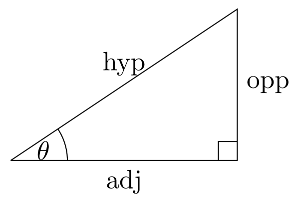
Tangent, secant, cosecant, and cotangent

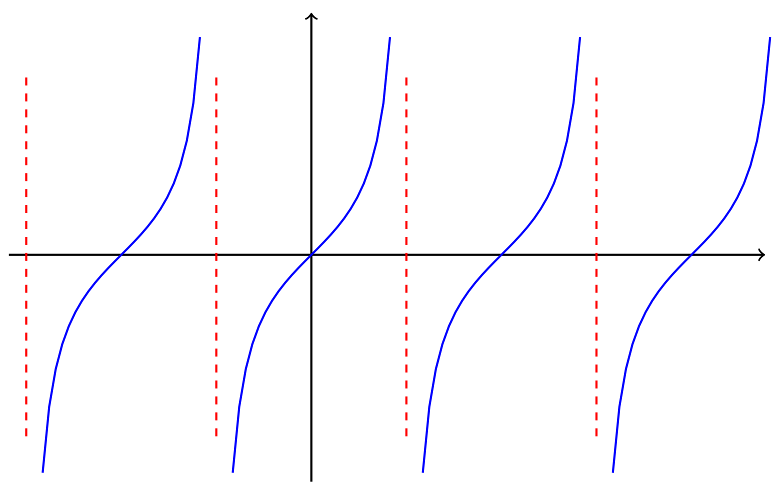
The two most important trig functions are sine and cosine. There are four other trig functions that are built off of them. Of them, the most important is the *tangent* function. The tangent is defined in either of the following two ways:

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad\qquad \tan\theta = \frac{\text{opp}}{\text{adj}}.$$



For example, since $\sin(\pi/4)$ and $\cos\pi/2$ are both $\sqrt{2}/2$, we have $\tan(\pi/4) = \frac{\sqrt{2}/2}{\sqrt{2}/2} = 1$. Or, working directly off the 45-45-90 triangle, we get that $\tan(\pi/4) = 1/1 = 1$ since both the opposite and adjacent sides are 1.

Here is the graph of $\tan x$:



Notice the asymptotes. They happen at odd multiples of $\pi/2$, namely $\pi/2$, $3\pi/2$, $5\pi/2$, etc. and their negatives. These are precisely the places that the cosine (denominator of the tangent) is 0.

The other three trig functions are the *secant*, *cosecant*, and *cotangent*. They are defined as below:

$$\sec\theta = \frac{1}{\cos\theta} \qquad \sec\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{\cos\theta}{\sin\theta}$$

We don't include their graphs or right triangle definitions here. They are nice to know, but not critical. One thing that is critically important is how to rewrite trig expressions using the rules above. Here are a couple of examples:

Example 1 Rewrite $\dfrac{\sec x}{\tan x}$ in terms of sines and cosines, simplifying as much as possible.

Solution: We have

$$\frac{\sec x}{\tan x} = \frac{1/\cos x}{\sin x/\cos x}.$$

Multiply top and bottom by $\cos x/\sin x$ to simplify the complex fraction, and we get the above equal to $1/\sin x$, which we could also write as $\csc x$.

Example 2 Show that $\cos x/\sin^2 x = \csc x \cot x$.

Solution:: Break up the numerator and denominator like below:

$$\frac{\cos x}{\sin^2 x} = \frac{1 \cdot \cos x}{\sin x \cdot \sin x} = \frac{1}{\sin x} \cdot \frac{\cos x}{\sin x} = \csc x \cot x.$$

Exercises

1. Find the following without a calculator.
   (a) $\tan(\pi)$
   (b) $\sec(2\pi)$
   (c) $\csc(\pi/6)$
   (d) $\tan(3\pi/4)$
2. Use a calculator to estimate the following to 4 decimal places.
   (a) $\tan(2.03)$
   (b) $\csc(\pi/11)$
3. Rewrite the following so that they are in terms of sines and cosines only, simplifying as much as possible.
   (a) $\sec^2 x \cot x$
   (b) $\dfrac{\csc x}{\tan x}$
4. Show the following are true using the rules for secant, cosecant, and cotangent.
   (a) $\dfrac{\sin^2 x}{\cos^3 x} = \tan^2 x \sec x$
   (b) $\dfrac{\sin x \csc x}{\cos x} = \sec x$

Answers

1. (a) $\tan(\pi) = 0$ — The angle π corresponds to $(-1, 0)$ on the unit circle. So $\cos(\pi) = -1$ and $\sin(\pi) = 0$, making $\tan(\pi) = 0/-1 = 0$.

   (b) $\sec(2\pi) = 1$ — The angle 2π is the same as 0, which is at the point $(1, 0)$ on the unit circle. So $\cos(2\pi) = 1$ and $\sec(2\pi) = 1/1 = 1$.

   (c) $\csc(\pi/6) = 2$ — We know that $\sin(\pi/6) = 1/2$, so $\csc(\pi/6) = 1/(1/2) = 2$.

   (d) $\tan(3\pi/4) = -1$ — The angle $3\pi/4$ is the same as $135°$. It corresponds to the point $(-\sqrt{2}/2, \sqrt{2}/2$ in the second quadrant of the unit circle. So $\sin(3\pi/4) = \sqrt{2}/2$, $\cos(3\pi/4) = -\sqrt{2}/2$, and dividing these gives the answer, which is -1.

2. (a) $\tan(2.03) = -2.0224$

   (b) $\csc(\pi/11) = 3.5495$

3. (a) $\sec^2 x \cot x = \frac{1}{\cos^2 x} \frac{\cos x}{\sin x} = \frac{1}{\cos x \sin x}$, where we cancel the cosine in the numerator with one of the cosines in the denominator.

   (b) $$\frac{\csc x}{\tan x} = \frac{\frac{1}{\sin x}}{\frac{\sin x}{\cos x}} \cdot \frac{\frac{\cos x}{\sin x}}{\frac{\cos x}{\sin x}} = \frac{\cos x}{\sin^2 x}$$

4. (a) One approach is to use the rules $\tan x = \frac{\sin x}{\cos x}$ and $\sec x = \frac{1}{\cos x}$ to rewrite the right side as $\frac{\sin^2 x}{\cos^2 x} \cdot \frac{1}{\cos x}$, and then multiply the fractions to get it equal to the left side.

   (b) Using $\csc x = \frac{1}{\sin x}$, the left side becomes $\frac{\sin x (1/sinx)}{\cos x}$ and the numerator simplifies to 1, leaving us with $\frac{1}{\cos x}$, which is the same as $\sec x$, the right side.