Solving other kinds of equations

Dealing with fractions

When solving equations such as $\frac{2}{x} = \frac{x}{3}$ or $\frac{x}{x+3} = 0$, we usually want to do something to clear the fraction, which usually involves multiplying the equation through by the denominator of one or both sides.

Example 1 Solve $\frac{2}{x} = \frac{x}{3}$ for x.

Solution: What we want to do is *cross-multiply* to get $6 = x^2$. We get these by multiplying each side by x and by 3, the two denominators. Another way to see this is you simply multiply each side by the other's denominator and get rid of the denominators. Once we have $6 = x^2$, we can solve this to get $x = \pm\sqrt{6}$.

Example 2 Solve $\frac{x+1}{x+3} = 0$ for x.

Solution: Multiply both sides by $x+3$ to clear the fraction. This leaves $x+1 = 0$, so our solution is $x = -1$.

Example 3 Solve $\frac{3}{x} = 0$ for x.

Solution: Multiply both sides by x to clear the fraction. This gives $3 = 0$. This is a nonsense statement, so there is no solution.

In general, multiplying by the denominator is the thing to do in these situations, but you need to make sure that the denominator and numerator aren't both 0 at the same point, as in the next example.

Example 4 Solve $\frac{x+1}{x^2+2x+3} = 0$ for x.

Solution: It would seem like we could multiply both sides by the denominator to get $x+1 = 0$ and then $x = -1$, but there is a problem. If we plug -1 into the original equation, we actually get $0/0$, which means the function is undefined at $x = -1$.

The problem comes because the denominator factors into $(x+1)(x+2)$. The better thing to do here would be to cancel the $x+1$ from the numerator and denominator to get $\frac{1}{x+2} = 0$. Then multiplying by $x+2$ to clear the fraction gives $1 = 0$, showing that there is no solution.

In general, multiplying through by the denominator usually works, but be careful to make sure that the numerator and denominator don't both turn out to be 0 at the same point.

Example 5 Solve $2x - \frac{100}{x^2} = 0$ for x.

Solution: Start by adding $\frac{100}{x^2}$ to both sides to get $2x = \frac{100}{x^2}$. Then multiply both sides by x^2 to clear the fraction. This gives $2x^3 = 100$. Divide both sides by 2 and take the cube root to get $x = \sqrt[3]{100}$.

Example 6 Solve $\frac{x}{2} - 1 = \frac{3}{x}$ for x.

Solution: One approach is to find a common denominator for the left side. This turns the equation into $\frac{x-2}{2} = \frac{3}{x}$. Then cross-multiply to get $x(x-2) = 6$. Distribute and move the 6 over to turn this into $x^2 - 2x - 6 = 0$. Finally, use the quadratic equation to get $\frac{2\pm\sqrt{4-24}}{2}$. Since the term in the radical is negative, these solutions are imaginary, so there is no real solution.

Exercises

Solve the following for x.

1. $\dfrac{3}{2x} = \dfrac{x}{5}$
2. $\dfrac{x^2 - 1}{2x + 3} = 0$
3. $\dfrac{x^2 + 1}{4x^2 + 2x + 3} = 0$
4. $4x^2 - \dfrac{2}{x^2} = 0$
5. $\dfrac{x}{2} + \dfrac{3}{x} = 1$

Answers

1. Cross-multiply to get $15 = 10x$. Then the solution is $x = 15/10$, which simplifies to $x = 3/2$.

2. Cross-multiply to get $x^2 - 1 = 0$. Move the 1 over and take the square root of both sides to get $x = \pm 1$.

3. Cross-multiply to get $x^2 + 1 = 0$. This has no real solution as $x^2 + 1$ is always greater than 0.

4. Multiply the first term by x^2/x^2 to find a common denominator. This gives $\frac{4x^4-2}{x^2} = 0$. Cross-multiply to get $4x^4 - 2 = 0$. This simplifies to $x^4 = 2$ from which we get $x = \pm\sqrt[4]{2}$.

5. To get a common denominator, multiply the first term by x/x and multiply the second by $2/2$. This gives $\frac{x^2+6}{2x} = 1$. Cross-multiply to get $x^2 + 6 = 2x$. Moving terms around gives $x^2 - 2x + 6 = 0$. Use the quadratic formula to get $x = \frac{2\pm\sqrt{-20}}{2}$, which are not real numbers. So there are no real solutions.