Solving higher degree polynomial equations

Simple equations

Solving equations of the form $x^n = a$ is very similar to solving quadratics like $x^2 = 4$. Just take roots, being careful about the $\pm$. In general, even roots need the $\pm$, but odd roots don't.

Example 1 Solve $x^3 = 11$ and $x^4 = 11$ for x.

Solution: For $x^3 = 11$, the solution is $\sqrt[3]{11}$ and for $x^4 = 11$, the solution is $\pm\sqrt[4]{11}$.

Factoring

For a lot of the types of problems that show up in calculus, the solution is to move everything to one side, factor as much as possible, and then set each factor to 0 separately. Here are several examples.

Example 1 Solve $x^2(x-1) + 3x(x-1)^2 = 0$ for x.

Solution: First factor the left side into $x(x-1)[x+3(x-1)]$. This simplifies into $x(x-1)(4x-3)$.

Then set each part equal to 0 separately to get $x = 0$, $x - 1 = 0$, and $4x - 3 = 0$. This then gives three solutions, $x = 0$, $x = 1$, and $x = 3/4$.

Example 2 Solve $4x^3 = 4x$ for x.

Solution: First move everything to one side to get $4x^3 - 4x = 0$. Then factor out a $4x$ to get $4x(x^2 - 1) = 0$. Then set each part to 0 separately to get $4x = 0$ and $x^2 - 1 = 0$. Then solve each of these to get $x = 0$ and $x = 1$, and $x = -1$.

Example 3 Solve $x^4 \cdot 3(x-1)^2 + (x-1)^3 \cdot 4x^3 = 0$ for x.

Solution: We can factor x^3 and $(x-1)^2$ out of each term. This leaves us with $x^3(x-1)^2[3x+4(x-1)] = 0$. Simplify this into $x^3(x-1)^2(7x-4) = 0$. Then set each part equal to 0 separately. This gives us $x^3 = 0$, $(x-1)^2 = 0$, and $7x - 4 = 0$. Solve each of these to get $x = 0$, $x = 1$, and $x = 4/7$.

Example 4 Solve $12t^3 + 36t^2 + 24t = 0$.

Solution: Start by factoring $12t$ out to get $12t(t^2 + 3t + 2) = 0$. We can further factor the quadratic to get $12t(t+2)(t+1) = 0$. Set each part equal to 0 separately to get $12t = 0$, $t + 2 = 0$, and $t + 1 = 0$. This gives $t = 0$, $t = -2$, and $t = -1$.

Exercises

Solve the following for x.

1. $x^5 = 11$
2. $x^6 = 64$
3. $3x^3 - 2x = 0$
4. $4x(x-1)^2 + 8x^2(x-1)^2 = 0$
5. $x^2(x-1)^2 + 2x(x-1)^2 + (x-1)^2 = 0$

Answers

1. Take the fifth root of both sides to get $x = \sqrt[5]{11}$

2. Take the sixth root of both sides to get $x = \pm\sqrt[6]{64} = \pm 2$

3. Factor out an x to get $x(3x^2 - 2) = 0$. Set $x = 0$ and $3x^2 - 2 = 0$. The second equation becomes $3x^2 = 2$ and then $x^2 = 2/3$, giving $x = \pm\sqrt{2/3}$. So overall, the three solutions are $x = 0, \pm\sqrt{2/3}$

4. Factor it to get $4x(x-1)^2(1+2x) = 0$. Set all the components to 0 separately to get $4x = 0$, $(x-1)^2 = 0$ and $1 + 2x = 0$¿ The first equation gives $x = 0$. The second gives $x = 1$, and the third gives $x = -1/2$. So the three solutions are $x = 0, 1, -1/2$.

5. Factor it to get $(x-1)^2(x^2+2x+1) = 0$. Factor the quadratic to get $(x-1)^2(x+1)^2 = 0$. Set the two parts equal to 0 separately to get $(x-1)^2 = 0$ and $(x+1)^2 = 0$. The first equation gives $x = 1$ and the second gives $x = -1$. So the two solutions are $x = 1$ and $x = -1$.