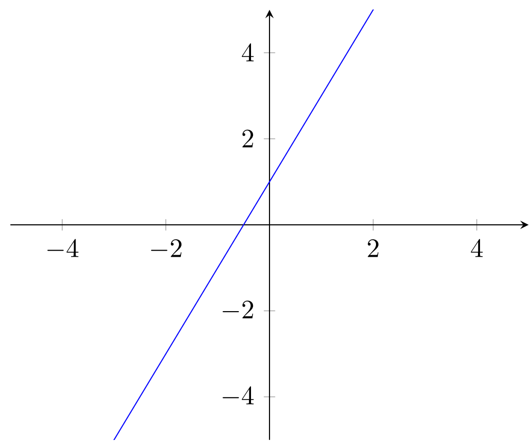
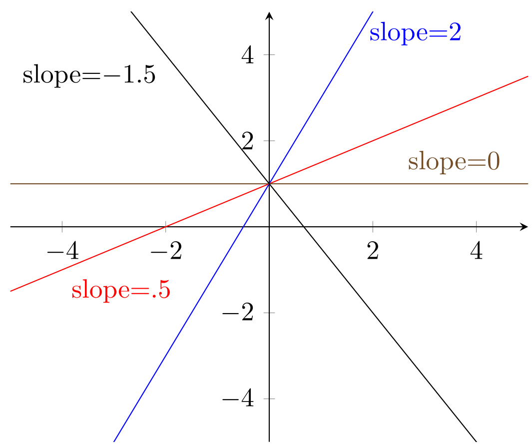
Lines

Shown below is the graph of $y = 2x + 1$.



In the equation $y = 2x + 1$, 2 is the slope and 1 is the y-intercept. The slope tells how steep the line is, and the y-intercept tells where the line crosses the y-axis. In this case it crosses at $y = 1$.

Slope is often described as “rise over run.” A slope of 2 means that the line rises 2 units up in the y-direction for every 1 unit of change in the x-direction. Shown below are several lines with varying slopes.



Notice in particular that lines with negative slopes go downward. Perfectly horizontal lines have a slope of 0. Their equations are always of the form $y = a$, where a can be any number (like $y = 1$ or $y = 7.52$). Perfectly vertical lines have an undefined (essentially infinite) slope. Their equations are always of the form $x = a$, where a can be any number.

Given two points, we can find the slope of the line between them using the rise over run idea. If the points are (x_1, y_1) and (x_2, y_2), the slope is given by

$$\frac{y_2 - y_1}{x_2 - x_1}.$$

For instance, the slope of the line between $(1, 2)$ and $(-3, 4)$ is

$$\frac{2 - 4}{1 - -3} = -\frac{1}{2}.$$

Note that if we reverse the order of the points to $(-3, 4)$ and $(1, 2)$, the slope would come out exactly the same, so we don’t have to worry about the order; just be consistent between the numerator and denominator.

Slope-intercept equation of a line

There are several ways of expressing the equation of a line. One of the most common and useful is the *slope-intercept* form, $y = mx + b$. Here m is the slope and b is the y-intercept.

Example Find the slope-intercept form of the line that passes through $(3, 5)$ and $(2, 9)$.

Solution: First, find the slope m using the rise over run formula:

$$m = \frac{9-5}{2-3} = -4.$$

So the line formula is $y = -4x + b$. We need to find b. To do so pick either of the points, plug it into the formula, and solve for b. Let's use the first point $(3, 5)$. Plugging in $x = 3$ and $y = 5$ gives $5 = -4(3) + b$. Solve to get $b = 17$. Therefore, the line's formula is $y = -4x + 17$.

Point-slope equation of a line

Another useful way of expressing the equation of a line is the *point-slope* form, $y - y_0 = m(x - x_0)$. Here m is the slope and (x_0, y_0) can be any point on the line. This is a particularly useful formula in calculus when you already know the slope of a line.

Example Find the point-slope form of the line with slope 3 that passes through $(2, 5)$.

Solution: Using the point-slope form, the line's equation is

$$y - 5 = 3(x - 2).$$

If needed, it's not too hard to rewrite this in slope-intercept form by simplifying. Distribute the 3 on the right side and add 5 to both sides to get $y = 3x - 1$.

Real-life meaning of slope

Line equations are often used to represent real things. For instance, $y = -.05x + 4.96$ might represent the amount of time people spend per day using a cell phone, given their age. Specifically, if you are x years old, then y predicts how many hours a day you typically use a cell phone. For instance, if $x = 40$, it predicts $-.05(40) + 4.96 = 2.96$ hours.

The slope -.05 here has a particular meaning. If there is a change of 1 unit in the x-direction, the slope tells us how much of a change there is in the y-direction. Here, the slope of -.05 tells us that for every 1 additional year of age, cell phone use drops by .05 hours.

Exercises

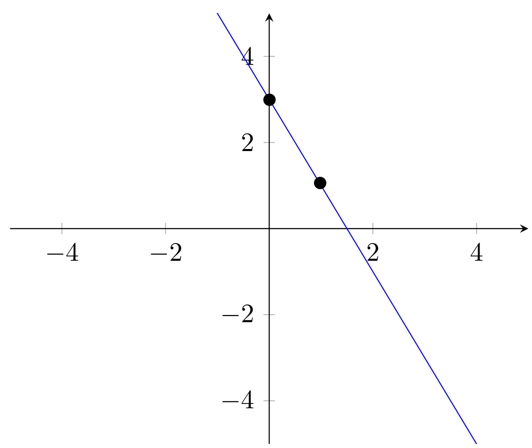
1. A line has equation $y = 3x - 7$. What are its slope and y-intercept?
2. What is the slope of the line passing through the points $(2, 3)$ and $(4, 7)$?
3. Find the equation of the line passing through the points $(-2, 3)$ and $(5, 9)$.
4. Find the equation of the line passing through the point $(5, 6)$ with a slope of 2.
5. What is the equation of the vertical line that passes through the point $(2, 5)$?
6. What is the equation of the horizontal line that passes through the point $(2, 5)$?
7. Sketch the line $y = -2x + 3$
8. Sketch the line $y - 3 = \frac{1}{2}(x - 4)$.
9. The cost of producing x units of a product is estimated by the line equation $y = 41.5x + 215$, where x is the number of items of the product made. Give an interpretation of the the slope, 41.5, in real-world terms.

Answers

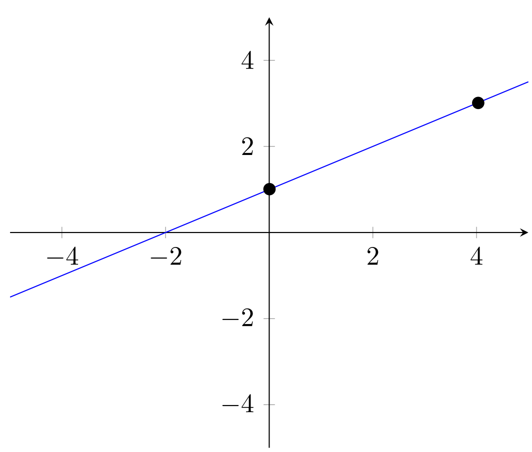
1. The slope is 3 and the intercept is -7.
2. The slope is $\frac{7-3}{4-2} = 2$.
3. The slope is $\frac{9-3}{5--2} = \frac{6}{7}$.

   Using the slope-intercept formula, we have $y = \frac{6}{7}x + b$. Plug in the point $(-2, 3)$ to get $3 = \frac{6}{7}(-2) + b$ and solve to get $b = \frac{30}{7}$. So the equation is $y = \frac{6}{7} + \frac{30}{7}$.

   We could also do this with the point-slope form. Using the point $(-2, 3)$, the point-slope form is $y - 3 = \frac{6}{7}(x + 2)$.
4. The point-slope form for this is $y - 6 = 2(x - 5)$. We could rewrite this as $y = 2x - 4$.
5. $x = 2$. Vertical lines are always of the form $x =$ something, and that something must be 2 here since the line passes through $(2, 5)$.
6. $y = 5$. Horizontal lines are always of the form $y =$ something, and that something must be 5 here since the line passes through $(2, 5)$.
7. Pick any two values of x, find the y-coordinates, and draw a line between the two points. For instance, if we take $x = 0$, we get $y = 3$, and if we take $x = 1$, we get $y = 1$. So we draw a line between $(0, 3)$ and $(1, 1)$.



8. Pick any two values of x, like $x = 0$ and $x = 4$, and find their y-values. For $x = 0$, we get $y = 1$, and for $x = 4$, we get $y = 3$. Plot the points $(0, 1)$ and $(4, 3)$ and connect them with a line.



9. Each additional item that is produced adds 41.5 to the profit.