Domain and range

Domain

The domain of a function is the set of values that you can plug into the function. Often what is most interesting about the domain is not what you can plug in to a function, but what you can't.

To find the domain, think about things that are not allowed: dividing by 0 and taking square roots of a negative number. Usually in calculus, the places these happen are not in the domain and everything else is.

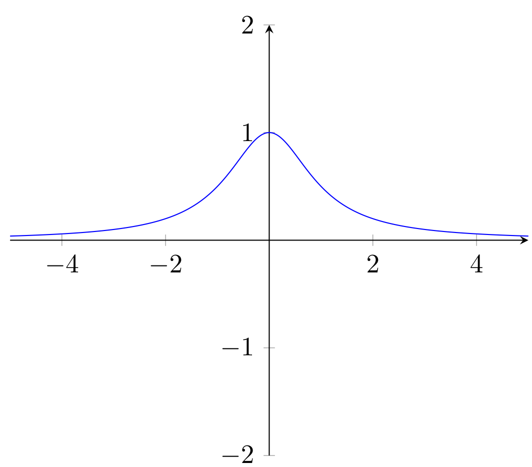
Here are a few examples:

1. If $f(x) = \frac{1}{x}$, you can plug in any number except 0. So the domain is all real numbers except 0. People often write this as $(-\infty, 0) \cup (0, \infty)$
2. To find the domain of $f(x) = \frac{x^2+1}{x^2-1}$, look at where the denominator is 0. So set $x^2 - 1 = 0$ and solve to get $x = 1$ and -1. So domain is all real numbers except -1 and 1.
3. If $f(x) = \sqrt{x}$, you can only plug in positive numbers or 0. Negatives are not allowed. So the domain is all nonnegative real numbers, or $[0, \infty)$
4. If $f(x) = \sqrt{x-3}$, we are okay as long as we don't have a negative in the radical. This means that $x - 3 \geq 0$, so $x \geq 3$. So the domain is all values of x greater than or equal to 3, or $[3, \infty)$
5. If $f(x) = x^2 + 3x + 9$, the domain is all real numbers. There is nothing that can go wrong by plugging in a real number. In fact, the domain of any polynomial is all real numbers.

Range

The range of a function is all the possible outputs. When looking at the graph, it's all the possible y-values. Here are a few examples:

1. The range of $f(x) = x^2$ is all nonnegative real numbers, or $[0, \infty)$. When squaring a number we can't get a negative, but we can get any positive number or 0.
2. The range of $f(x) = \dfrac{1}{x^2+1}$ is $(0, 1]$ as can be seen from the graph below.



   The peak of the function occurs at $x = 1$ and it has a horizontal asymptote at $y = 0$. The function never quite reaches 0.

3. The range of $f(x) = x$ is all real numbers. Every real number is a possible output.

Exercises

1. Find the domain of the following functions.

   (a) $f(x) = \dfrac{x}{x-3}$

   (b) $f(x) = \dfrac{2+x}{x^2+4x-5}$

   (c) $f(x) = \sqrt{2-x}$

   (d) $f(x) = \sqrt{x^2+1}$

   (e) $f(x) = x^4 + 4x^3 + 3x^2 + 2x + 9$

2. Find the range of the following functions.

   (a) $f(x) = x^3$

   (b) $f(x) = \dfrac{1}{x^2}$

Answers

1. (a) The only thing that could go wrong with this function is if the denominator is 0. That happens when $x-3=0$, which is when $x=3$. So the domain is all real numbers except 3. $f(x)=\dfrac{x}{x-3}$

   (b) The only thing that could go wrong with this function is if the denominator is 0. That happens when $x^2+4x-5=0$. Factoring the left side, we get $(x+5)(x-1)$. Setting these equal to 0 gives $x+5=0$ and $x-1=0$, which then gives $x=-5$ and $x=1$. The domain is all real numbers except -5 and 1.

   (c) The expression under the radical needs to be positive. Thus we need $2-x\geq 0$, which means $2\geq x$ or $x\leq 2$. So the domain is all real numbers less than or equal to 2.

   (d) The expression under the radical needs to be positive. However, that expression, x^2+1, is always positive, so there are no problems for this function. It is defined for all real numbers.

   (e) There is nothing that can go wrong with this function. It is a polynomial. So the domain is all real numbers.

2. (a) The range is all real numbers. Every real number is the cube of some other number, namely it's cube root.

   (b) The range is all positive real numbers. Because x^2 and 1 are both positive, it is impossible to get a negative out of this, and setting $1/x^2=0$ gives $1=0$, so the function can never be 0. On the other, hand, any positive number is a possible output of this function. In particular, for any positive number r, if we take $x=1/\sqrt{r}$, then $1/x^2=r$.