Basic factoring

General rules

Factoring is like doing the distributive rule in reverse. To factor, look at all the terms and see what they all have in common. Then you pull that least common factor out of each term.

For instance, consider the expression $9x^3 + 12x^2$. First, look at the coefficients 9 and 12. They have a common factor of 3. Then look at the x terms. They have a common factor of x^2. So we can factor out a 3 and an x^2. This turns the expression into

$$3x^2(3x + 4).$$

To get the $3x + 4$ part, what we do is first look at the $9x^3$ and ask what is left over after pulling out $3x^2$. Then we look at the $12x^2$ term and ask the same question.

To check our work, we could use the distributive rule on $3x^2(3x + 4)$ to show that it really does equal the original expression $9x^3 + 12x^2$.

Here are several more examples:

Example 1 Factor $4x^2 + 10x$.

Solution: The coefficients 4 and 10 each have a factor of 2 in common. The x terms each have a factor of x in common. So we will factor out $2x$.

Pulling out $2x$ from $4x^2$ leaves $2x$. Pulling out $2x$ from $10x$ leaves 5. So the expression factors into

$$2x(2x + 5).$$

Example 2 Factor $x^4 + 5x^3 + x^2$.

Solution: The coefficients 1, 5, and 1 don't have any factor greater than 1 in common, so we won't be able to pull out any constant factors. The x terms, however all have an x^2 in common.

Pulling an x^2 out of x^4 leaves x^2, pulling x^2 out of $5x^3$ leaves $5x$, and pulling x^2 out of x^2 leaves just a 1. So the expression factors into

$$x^2(x^2 + 5x + 1).$$

Example 3 Factor $3x^2(x - 1) + 2(x - 1)^2$.

Solution: The coefficients 3 and 2 don't have anything in common. The first term contains an x^2, but the second has no x term, so we can't factor any x terms out. However, the first term and the second term do share an $(x - 1)$ term, so we can factor that out, like below:

$$(x - 1)[3x^2 + 2(x - 1)].$$

Example 4 Factor $3x(x - 1)^2(x - 2) + 6x^3(x - 1)^3$.

Solution: First, the coefficients 3 and 6 both contain a factor of 3. Next, looking at the x terms, there is an x and x^3, so we can pull out an x from both terms. Next, looking at the $(x - 1)$ terms, we have $(x - 1)^2$ and $(x - 1)^3$, so we can pull out $(x - 1)^2$. Finally, the first term contains an $(x - 2)$, but the second term doesn't, so there's nothing we can do with that. The overall expression factors into

$$3x(x - 1)^2[(x - 2) + 2x(x - 1)].$$

Note that we could simplify the expression in brackets, but it's only worth doing so if we actually need it to be simplified.

Example 5 Factor $5xy^4 + 5x^2y^3 + 15x^2y^5$.

Solution: The coefficients, 5, 5, and 15 each contain a factor of 5. Next, the x terms are x, x^2, and x^2, so we can pull out an x. Finally, the y terms are y^4, y^3, and y^5, so we can pull out a y^3. The overall expression factors into

$$5xy^3(y + x + 3xy^2).$$

Example 6 Factor $\sqrt{x}(x-2) + 4\sqrt{x}$.

Solution: The coefficients are 1 and 4, so we don't get any constant terms to factor out. Each term does have a $\sqrt{x}$, so we can factor that out. Finally, the first term has an $(x-2)$ term, but the second doesn't, so we don't get anything from that. The overall expression factors into

$$\sqrt{x}(x - 2 + 4).$$

We can rewrite this as $\sqrt{x}(x+2)$.

Exercises

Factor the following as completely as possible.

1. $2x^2 + 4x$
2. $x^3 + 5x^2 + x$
3. $x(x-1)^2 + 3x(x-1)^3$
4. $4x^2y^3 + 5x^4y^3 + 6x^2y^5$
5. $2x(x-1)^2(x+2) + x(x-1)(x+2)^2$
6. $4\sqrt[3]{x} + 10\sqrt[3]{x}(x-1)$

Answers

1. $2x^2 + 4x = 2x(x+2)$

2. $x^3 + 5x^2 + x = x(x^2 + 5x + 1)$

3. $x(x-1)^2 + 3x(x-1)^3 = x(x-1)^2[1 + 3(x-1) = x(x-1)^2(3x-2)$

4. $4x^2y^3 + 5x^4y^3 + 6x^2y^5 = x^2y^3(4 + 5x^2y + 6y^2)$

5. $2x(x-1)^2(x+2) + x(x-1)(x+2)^2 =$

   $x(x-1)(x+2)[2(x-1) + (x+2)] =$

   $x(x-1)(x+2)[3x] =$

   $3x^2(x-1)(x+2)$

6. $4\sqrt[3]{x} + 10\sqrt[3]{x}(x-1) = 2\sqrt[3]{x}[2 + 5(x-1)] = 2\sqrt[3]{x}(x-3)$