The Mandelbrot Set

Brian Heinold

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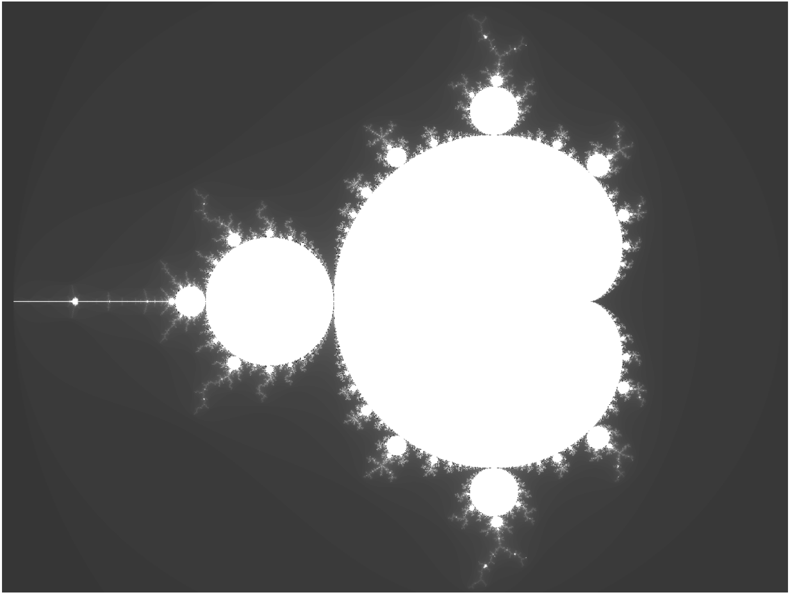

Benoit Mandelbrot

Discovered by Benoit Mandelbrot (1924-2010) in the 1970s while working at IBM.

Mandelbrot's original image

Iteration

Example: Let $f(x) = x^2$ and start with $x = 2$.

$f(2) = 4$
$f(4) = 16$
$f(16) = 256$
$f(256) = 65536$
$\dots$

Iterates are approaching ∞.

A different starting point

Let $f(x) = x^2$ and start with $x = \frac{1}{2}$.

$f(\frac{1}{2}) = \frac{1}{4}$

$f(\frac{1}{4}) = \frac{1}{16}$

$f(\frac{1}{16}) = \frac{1}{256}$

$f(\frac{1}{256}) = \frac{1}{65536}$

$\ldots$

Iterates are approaching 0.

Another example

Let $f(x) = -x$ and start with $x = 1$.

$f(1) = -1$
$f(-1) = 1$
$f(1) = -1$
$f(-1) = 1$
$\dots$

Iterates are not settling down on a value.

Coloring $f(x) = x^2$ by convergence

Color each point according to how fast it converges.

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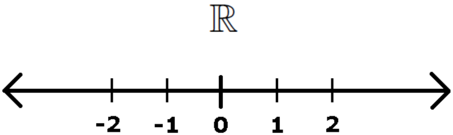
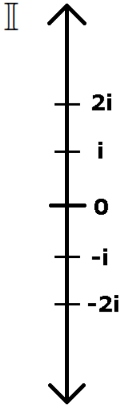
Interesting, but boring. We need to move to two dimensions!

Complex numbers

$i = \sqrt{-1}$

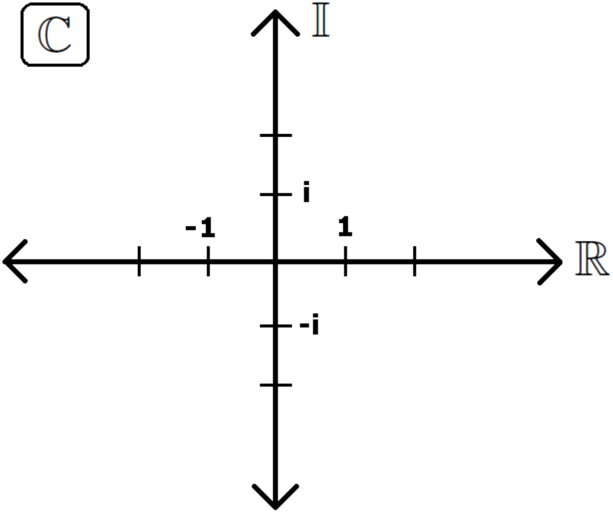
Complex numbers: $7i$, $2 + 3i$, $3.4 - 1.64i$, $\ldots$





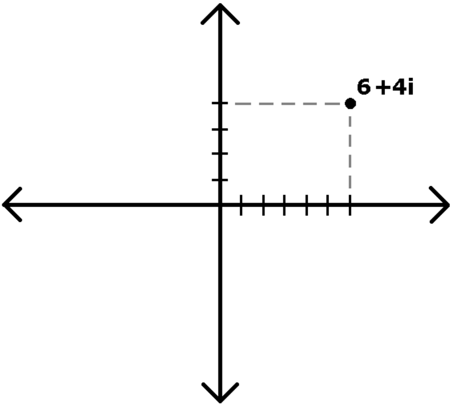
Picturing complex numbers

Put the reals and imaginaries together to get $\mathbb{C}$, the complex numbers.



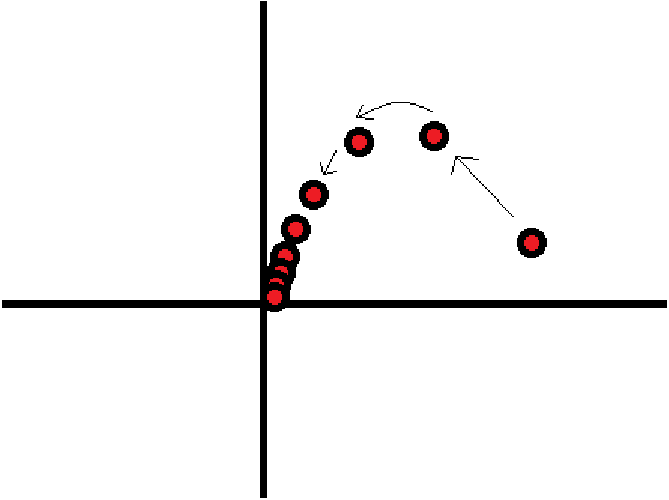
Picturing complex numbers

Every complex number is a combination of a real part and an imaginary part.



Iteration with complex numbers

Plug $z = x + iy$ into $f(z)$. Get a value, and plug that value into the function. Then plug the result of that into the function, etc.



Example of iteration with complex numbers

Consider $f(z) = z^2$ with $z = 1 + 2i$:

$$f(1+2i) = (1+2i)(1+2i) = -3+4i$$

$$f(-3+4i) = (-3+4i)(-3+4i) = -7-24i$$

$$f(-7-24i) = (-7-24i)(-7-24i) = -527+336i$$

Iterates are pretty clearly heading off to ∞.

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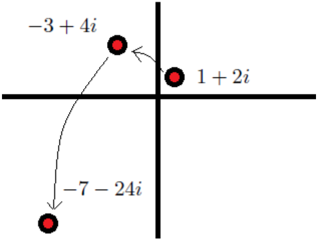

Iteration with complex numbers

Color points according to how fast they converge under $f(x) = x^2$ and $f(z) = z^2$.

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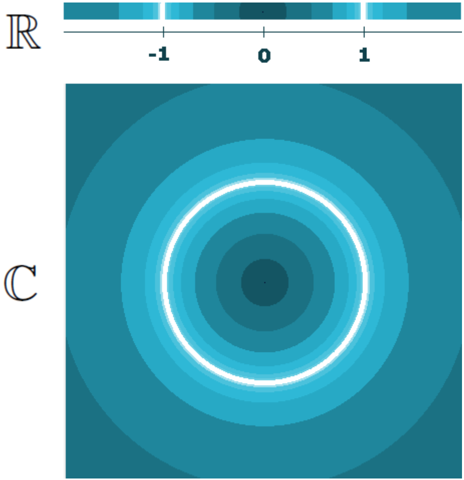

A small change

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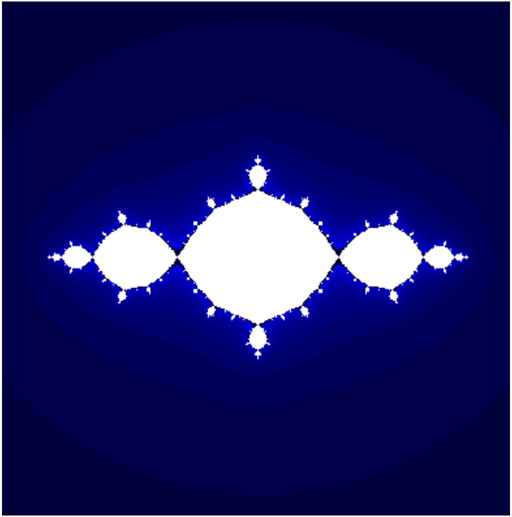
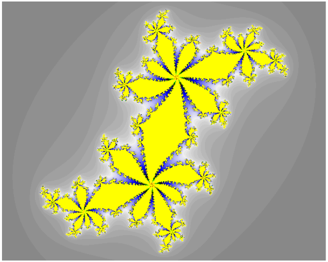
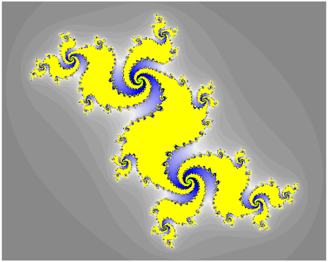


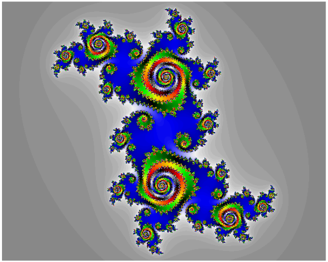
Iterating $z^2 + c$ for various values of c



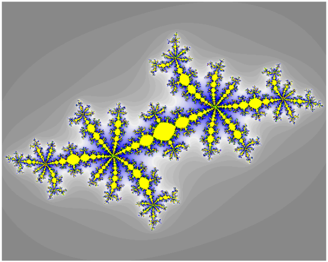
.12-.62i



-.06+.68i



.27+.49i



-.65-.44i

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- Is there a way we can tell what type of Julia set we will get for $f(z) = z^2 + c$, just by looking at what happens to a single starting point when iterated?
- Yes. What happens to 0 determines a lot about what the Julia set looks like.
- If we do this for lots of c values, and plot just what happens to 0, we get the Mandelbrot set.

Plotting the Mandelbrot set

It is usually plotted as follows:

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- If this happens, we color the point according to how many iterations it took to get there.
- Otherwise, color the point yellow (or white or whatever — just be consistent)

Programs

Time for some programs. . .

Thanks

Thank you for your attention.

Image credits

- First Mandelbrot set — `http://paulscottinfo.ipage.com/art-of-maths/4mandelbrot.html`
- Mandelbrot himself — `http://www.rugusavay.com/benoit-mandelbrot-photos/`