Smalltalk: Asking for too much

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September 14, 2018

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  How to handle the fractions of a seat?

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- Seems reasonable, right?

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- If you're okay with breaking the Quota rule, then you can avoid the two paradoxes, but if you insist on the Quota rule, then you can construct scenarios where the paradoxes will occur for your apportionment method.

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  3. *Independence of a third alternative* — The system's ranking of A versus B should not be affected by voter preferences for a third candidate C.
- In 1952, Nobel Laureate Kenneth Arrow proved that there is no ranked voting system that satisfies all of these conditions (if there are more than 2 candidates).

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- It also demonstrates the key idea of what follows — feeding the rule back into itself to derive a contradiction.

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- Either way, we have a contradiction.

Another logical paradox

- Question: In a logic class, your whole grade is based on a single statement you have to make. If you make a true statement, then you get a course grade of 20% and you fail the course. If you make a false statement, then you get a grade of 10% and you fail the course. What can you say?

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- This is a contradiction.
- Notice again how we are feeding the problem back into itself. Our statement references the problem itself. Terrence Tao describes this as the "no self-defeating object" argument.

Halting Problem

- Two functions:

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def f():
    i = 0
    while i < 10:
        i = i + 1
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def g():
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- Halting problem: Can we create a program that takes any function as an input and outputs whether or not that function eventually stops running?

- This is asking for too much. The existence of such a program would create a paradoxical situation.

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- If it says that h runs forever, the if statement would not catch, and h would halt.
- A contradiction.

Halting problem, continued

- The Halting problem is one of the most famous problems in computer science, showing there are definite limits on what can be computed.
- But it is a little abstract. Let's look at something more concrete.

The Post correspondence problem

- We are given the following two families of strings of 0s and 1s.

	A	B
1.	10	1
2.	010	01
3.	01	001

- Take a sequence of numbers, like (2, 2, 1), and add strings 2, 2, and 1 together strings from each family:
  - Family A: $01 + 010 + 10 = 0101010$
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- Can we find a sequence where things do work out to be equal?
- Yes. One example is (2, 3), which produces 01001 from both families.

The Post correspondence problem

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- I feel like I could program a solution. But this is asking for too much. One can prove that no such algorithm can exist.
- The proof works by constructing families of strings such that a solution to that family would give a solution to the Halting problem. The details are a little tricky.

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- Axioms are things that are taken for granted. They should be simple and there shouldn't be very many of them.
- In the early 1900s, there was a push to find a set of axioms from which all of mathematics could be derived.
- Prominent mathematicians worked on this for years. The most famous attempt was Bertrand Russell's and Alfred North Whitehead's *Principia Mathematica*.

An example set of axioms

Here are the famous Peano Axioms defining the natural numbers $\{1, 2, 3, \ldots\}$:

1. 1 is a natural number.
2. Whenever n is a natural number, the successor of n is also a natural number
3. 1 is not the successor of any natural number.
4. If the successors of n and m are equal, then $n = m$.
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From these we can derive all the familiar properties of natural numbers, such as $1 + 1 = 2$. We can build on this to derive other sets of numbers like all integers, rationals, and reals.

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- But in the early 1930s, Kurt Gödel proved that we can't have both. Roughly, any nontrivial set of axioms about arithmetic is either inconsistent or incomplete.
- This is known as Gödel's First incompleteness theorem.
- The proof of the theorem is tricky, but it relies on the same self-referential idea from the earlier logical paradoxes.

Incompleteness theorems, continued

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- I find it profoundly interesting — we can't reduce math to a mechanical system of rules. There is always something new out there.
- People wonder if certain famous unsolved problems in number theory, like the twin primes conjecture or Goldbach's conjecture, might actually be true statements that can't be proved using the standard axioms of mathematics.

This talk is incomplete

I've tried to cram far too much material into 30 minutes. Time to stop.

Thanks for your attention!