Is it all in your imagination?

Brian Heinold

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- Specifically, i is a number such that $i^2 = -1$.
- This is nonsensical. A number times itself must be positive, right?

What others are saying about them...

In 1545, Girolamo Cardano, who was the first to write about them, called them

"as subtle as they are useless"

In 1572, Rafael Bombelli, who developed the rules for working with them, said

"The whole matter seems to rest on sophistry rather than truth."

What others are saying about them...

In 1702 Gottfried von Leibniz, co-inventor of calculus, called i

"that amphibian between existence and nonexistence""

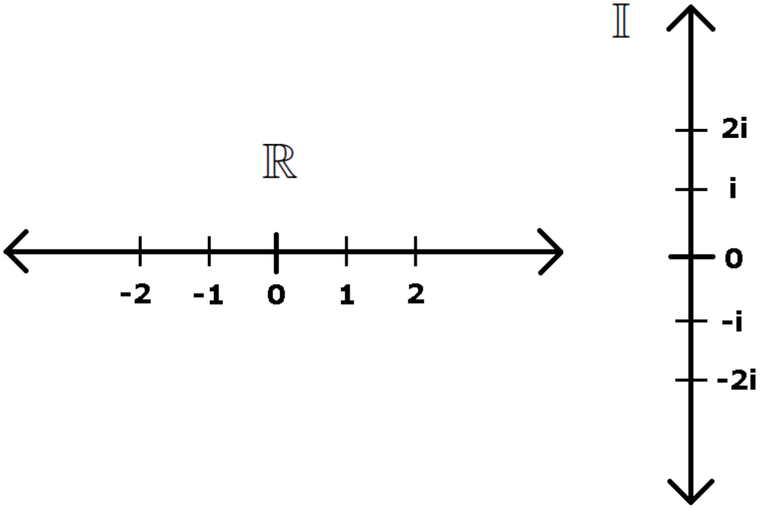
In 1770 Leonhard Euler, arguably the greatest mathematician of all time, wrote about

"numbers, which from their nature are impossible; and therefore they are usually called imaginary quantities, because they exist merely in the imagination. . .

Euler, continued. . .

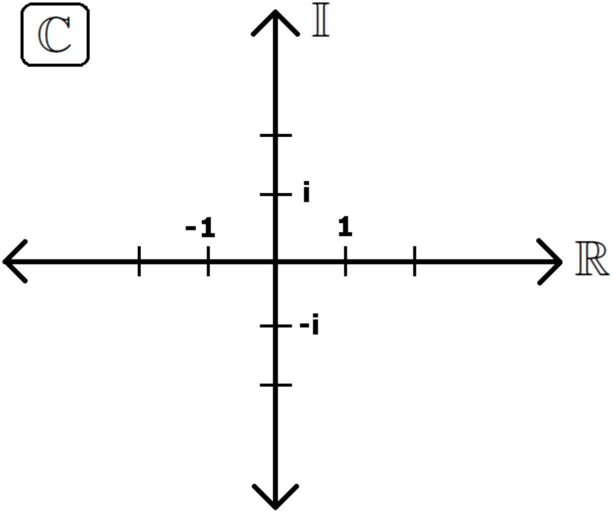
But notwithstanding this, these numbers present themselves to the mind; they exist in our imagination, and we still have a sufficient idea of them.

Picturing real and imaginary numbers



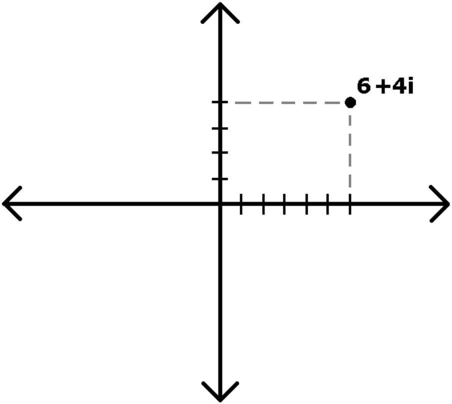
Picturing complex numbers

Put the reals and imaginaries together to get $\mathbb{C}$, the complex numbers.



Picturing complex numbers

Every complex number is a combination of a real part and an imaginary part.



Our numbers are two-dimensional now!

Working with complex numbers

Use the rules of algebra:

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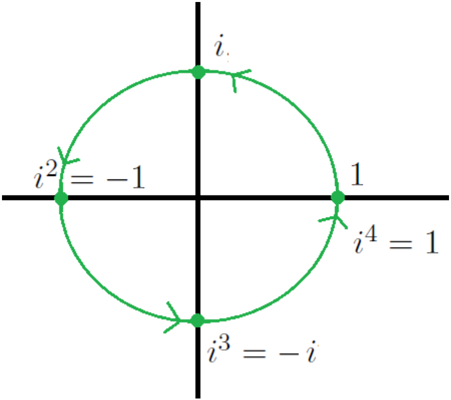
$$\frac{3+4i}{6+3i} = \frac{3+4i}{6+3i}\cdot\frac{6-3i}{6-3i} = \frac{30+12i}{25} = \frac{6}{5}+\frac{12}{25}i$$

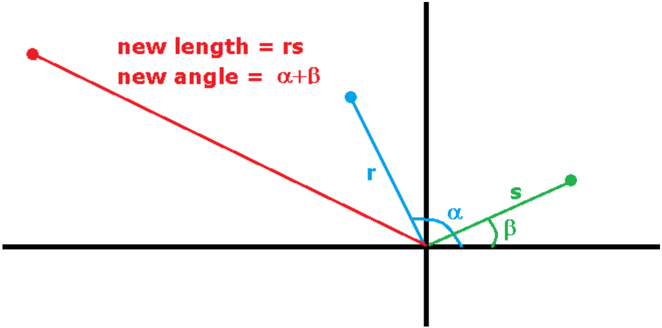
Rotation

Multiplication by i corresponds to rotation by 90°.



Rotation

In general, multiplying two complex numbers corresponds to adding their angles and multiplying their lengths.



Applications

- Complex numbers are applicable in places where rotation naturally fits.
- There are a number of such places in physics where complex numbers considerably simplify things:
  - Electromagnetic field
    - electric portion — real part
    - magnetic portion — imaginary part
  - Electrical circuit
    - capacitance — real part
    - inductance — imaginary part

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- Complex analysis used to prove the Prime Number theorem (number of primes less than n is $\approx \frac{n}{\ln n}$).
- Jacques Hadamard (1865-1963): "the shortest path between two truths in the real domain passes through the complex domain."

Euler's formula

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Ties together some of the most important functions in math.

"The most remarkable formula in all of math":

$$e^{i\pi} + 1 = 0.$$

Proof that $e^{i\theta} = \cos\theta + i\sin\theta$

Taylor series:

$$
\begin{aligned}
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \\
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots \\
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots
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\begin{aligned}
e^{ix} &= 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \dots \\
&= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right)
\end{aligned}
$$

Interesting facts

- $e^{i\theta} = \cos\theta + i\sin\theta$ provides a compact way to represent waves and oscillations.

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- $i^i = (e^{i\pi/2})^i = e^{i^2\pi/2} = e^{-\pi/2} = .2078\ldots$

Complex numbers make their presence felt on the reals

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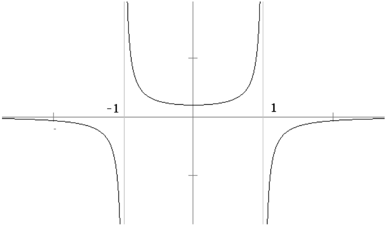
- This is because $\frac{1}{1-x^2}$ has vertical asymptotes at ± 1, which prevent the power series from working past them.



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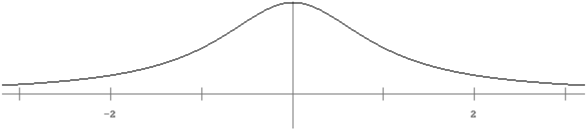

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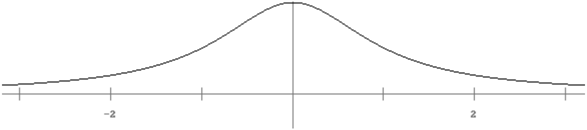
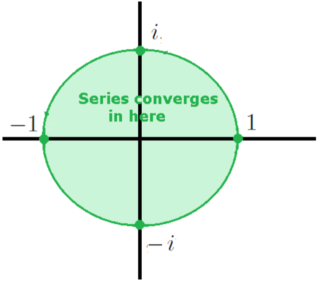


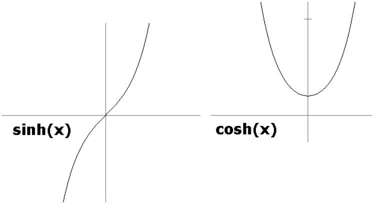
- The denominator has asymptotes at $\pm i$



So, how can imaginary numbers be imaginary if they have real effects?

Hyperbolic functions from calculus

$$\sinh x = \frac{e^x - e^{-x}}{2} \quad \cosh x = \frac{e^x + e^{-x}}{2}$$



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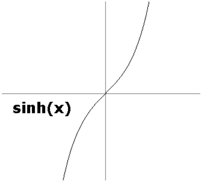

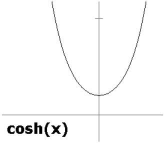

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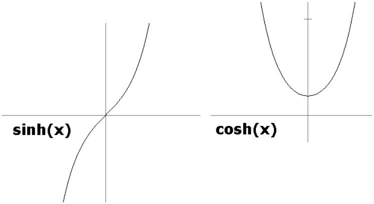

- Not periodic like $\sin x$ and $\cos x$.
- But $\frac{d}{dx}\sinh x = \cosh x$ and vice-versa.
- Also, they satisfy many of the same kinds of identities as ordinary trig functions:
    - $\sinh^2 x - \cosh^2 x = 1$
    - $\sinh(2x) = 2\sinh x \cosh x$
    - $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$

Complex sine and cosine

- From Euler's formula we get

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- In other words, sinh and cosh *are* periodic, just on the *imaginary* axis.
- On the imaginary axis, $\sin z$ and $\cos z$ behave like $\sinh z$ and $\cosh z$ on the real axis.

New things that we can solve

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- $\sin x = 3 \longrightarrow x = \dfrac{\pi}{2} + i \ln(3 + 2\sqrt{2})$
- $\ln(-1) = i\pi$

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The complex logarithm

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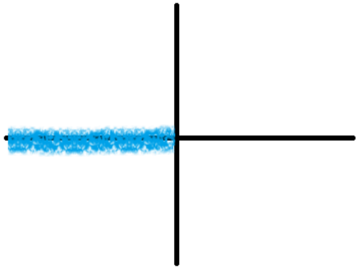
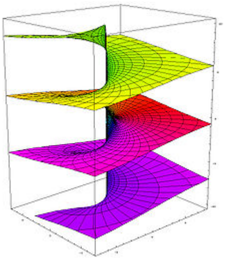
- e^z is periodic along the imaginary axis.
- So, $e^z = -1$ has infinitely many solutions: $e^{\pi i}, e^{2\pi i}, e^{3\pi i}, \ldots$
- This means $\log z$ is actually multivalued.



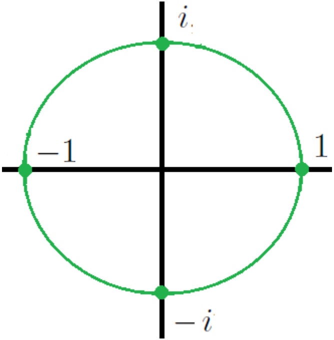


From http://www.answers.com/topic/branch-point

- This is an example of a *Riemann surface*

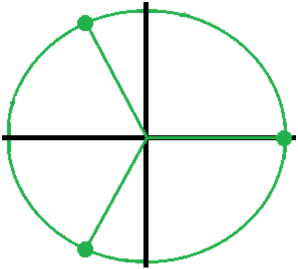
Roots of unity

- $x^2 = 1 \longrightarrow x = \pm 1$ (2 roots, 180° apart on unit circle)
- $x^4 = 1 \longrightarrow x = \pm 1, \pm i$ (4 roots, 90° apart on unit circle)



Roots of unity

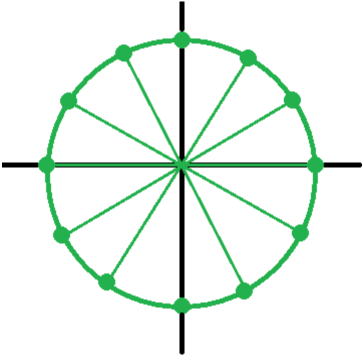
- What about $x^3 = 1$?
- 3 roots, spaced 120°($2\pi/3$ rad) apart on unit circle
- $x = 1$, $\cos(2\pi i/3) + \imath \sin(2\pi i/3)$, $\cos(4\pi/3) + \imath \sin(4\pi/3)$
- Can write as $x = e^{2\pi i k/3}$ for $k = 1, 2, 3$.



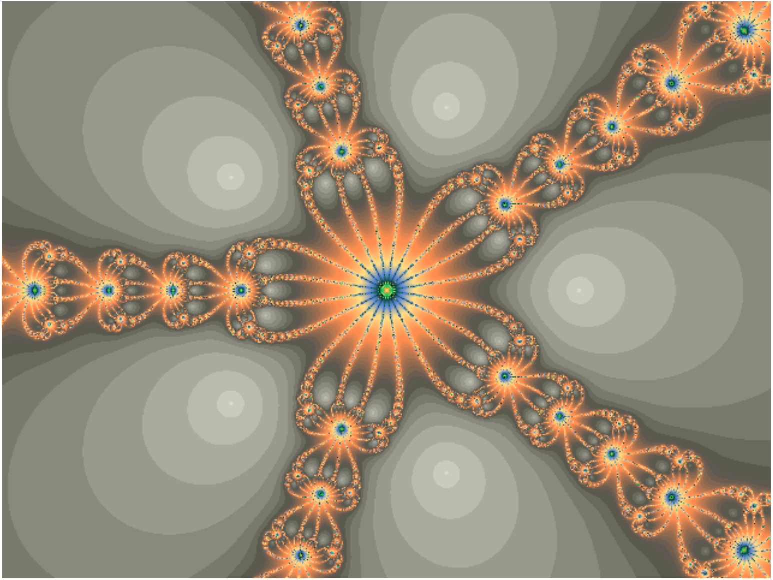
Roots of unity

In general, $x^n = 1$ has n roots, spaced $2\pi/n$ rad apart

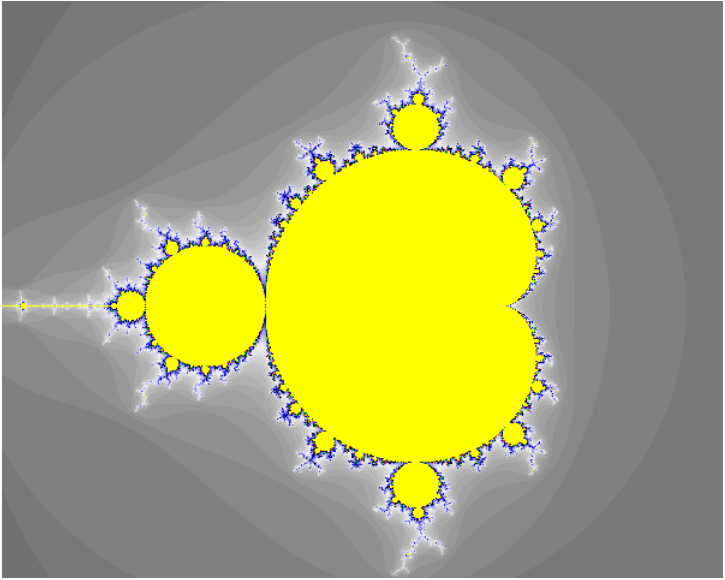
The roots are $e^{2\pi ik/n}$ for $k = 1, 2, \ldots, n$.



Using Newton's method to find the roots of unity



Mandelbrot set



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- Answer: Yes and no. Yes we can, and we can get things like the *quaternions* and *octonions*.
- But, no, we can't get anything as nice as $\mathbb{C}$. Adding dimensions causes you to lose nice properties like commutativity and associativity.
- So $\mathbb{C}$ is the largest as we can get without giving up things we'd rather not give up.

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  - There are -5 people in this room
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  - etc.

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  - There are -5 people in this room
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  - etc.
- But they are a natural fit for many other things:
  - Money: credit = + , debt = -
  - Motion: forward = +, backwards = -
  - etc.

Are fractions real?

- Fractions don't make sense for many things:
  - I have $\frac{2}{3}$ sisters.
  - There are $\frac{17}{19}$ books on my shelf.
  - etc.

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  - etc.
- But they are a natural fit for many other things:
  - I ate $\frac{1}{3}$ of a pizza
  - I walked $\frac{2}{3}$ of a mile
  - etc.

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- What exactly is the number 2 for instance?
- My answer: Complex numbers are as real as any other kind of number; they just don't appear in everyday life.

Thanks

Thank you for your attention.

Image credits

- Cardano – http://en.wikipedia.org/wiki/Gerolamo_Cardano
- Bombelli – http://www.learn-math.info/historyDetail.htm?id=Bombelli
- Leibniz – http://en.wikipedia.org/wiki/Gottfried_Wilhelm_Leibniz
- Euler – http://en.wikipedia.org/wiki/Leonhard_Euler