A few sum list coloring facts

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Berliner, Bostelmann, Brualdi, Deatt (BBBD) proved the following useful lemma.

Lemma 1 (BBBD Lemma 1)**.** *Let (G, f) be given, such that G is f-choosable. Suppose* $\text{size}(f) = \chi_{\text{SC}}(G) + r$ *for some $r \geq 0$. For any vertex $v \in V(G)$, and any set A of $r+1$ colors, there exists an f-assignment $\mathcal{C}$ such that every proper $\mathcal{C}$-coloring of G uses a color from A on v.*

Here is a slight strengthening of it.

Lemma 2. *Let (G, f) be given, with $v_0 \in V(G)$, and let $r \geq 0$. Define a size function g by $g(v) = f(v)$ for $v \neq v_0$ and $g(v_0) = f(v_0) - r - 1$. If G is not g-choosable, then for any set A of $r+1$ colors, there exists an f-assignment $\mathcal{C}$ such that every proper $\mathcal{C}$-coloring uses a color from A on v_0.*

Proof. Let $\mathcal{D}$ be an uncolorable g-assignment, with colors named so that $A \cap \mathcal{D}(v_0) = \emptyset$. Define the f-assignment $\mathcal{C}$ by $\mathcal{C}(v_0) = \mathcal{D}(v_0) \cup A$ and $\mathcal{C}(v) = \mathcal{D}(v)$ for $v \neq v_0$. □

The first lemma follows from this, since if $\text{size}(f) = \chi_{\text{SC}}(G) + r$, then g defined from f as in the second lemma has size less than $\chi_{\text{SC}}(G)$, and so G is not g-choosable. They proved the following nice theorem. Here's a modified version of their proof.

Lemma 3 (BBBD Theorem 1)**.** *Let G and G' be graphs such that $V(G) \cap V(G') = \{v_0\}$. Then*

$$\chi_{\text{SC}}(G \cup G') = \chi_{\text{SC}}(G) + \chi_{\text{SC}}(G') - 1.$$

Proof. To show $\chi_{\text{SC}}(G \cup G') \leq \chi_{\text{SC}}(G) + \chi_{\text{SC}}(G') - 1$, let g and g' be minimum choice functions on G and G', respectively. Define a function h on G of size $\chi_{\text{SC}}(G) + \chi_{\text{SC}}(G') - 1$ by $h(v_0) = g(v_0) + g'(v_0) - 1$, and let $h(v) = g(v)$ for $v \in V(G - v_0)$ and $h(v) = g'(v)$ for $v \in V(G' - v_0)$. Let $\mathcal{C}$ be an h-assignment. Because G is f choosable, and h agrees with f on G, except at v_0 where $h(v_0) \geq f(v_0)$, there exists a proper coloring of G from $\mathcal{C}_G$. Let A denote the set of all colors that can be used in a proper $\mathcal{C}_G$-coloring of G. If $|A| \geq g'(v)$, then there exists a proper $\mathcal{C}_{G'}$-coloring of G' with the color on v_0 coming from A, and hence this coloring can be combined with a proper $\mathcal{C}_G$-coloring of G to give a proper $\mathcal{C}$-coloring of $G \cup G'$. So we must show that we cannot have $|A| < g'(v)$. By way of contradiction, consider the list assignment $\mathcal{D}$ on G given by $\mathcal{D}(v) = \mathcal{C}(v)$ for $v \neq v_0$, and $\mathcal{D}(v_0) = \mathcal{C}(v_0) \setminus A$. Since $\mathcal{D}$ agrees on $G - v_0$ with $\mathcal{C}$ (which has a proper coloring), and $|\mathcal{D}(v_0)| = g(v_0) + g'(v_0) - 1 - |A| \geq g(v_0)$, there must be a proper coloring coloring from $\mathcal{D}$, which contradicts the definition of A.

Next, suppose there exists a choice function f of size $\chi_{\mathrm{SC}}(G) + \chi_{\mathrm{SC}}(G') - 2$. Since G is f-choosable, $\mathrm{size}(f_G) = \chi_{\mathrm{SC}}(G) + m$ for some $m \geq 0$. By BBBD's Lemma, there exists an f_G-assignment $\mathcal{C}$ such that the set A of colors that can be used in a proper $\mathcal{C}_G$-coloring of G has size at most $m + 1$. Define a size function h on G' by $h(v) = f(v)$ for $v \neq v_0$, and $h(v_0) = m + 1$. Then

$$\begin{aligned}
\mathrm{size}(h) &= m + 1 + \mathrm{size}(f_{G'-v}) \\
&= m + 1 + \mathrm{size}(f) - \mathrm{size}(f_G) \\
&= m + 1 + (\chi_{\mathrm{SC}}(G) + \chi_{\mathrm{SC}}(G') - 2) - (\chi_{\mathrm{SC}}(G) + m) \\
&= \chi_{\mathrm{SC}}(G') - 1.
\end{aligned}$$

Therefore, there must exist a h-assignment $\mathcal{D}$ on G' that has no proper coloring, and we may name the colors so that $\mathcal{D}(v_0) = A$. Then the f-assignment given by $\mathcal{C}$ on G and $\mathcal{D}$ on $G' - \{v_0\}$ has no proper coloring, contradicting that G is f-choosable. Hence, $\chi_{\mathrm{SC}}(G \cup G') \geq \chi_{\mathrm{SC}}(G) + \chi_{\mathrm{SC}}(G') - 1$. □

The basic idea of the above proof is that the f defined in the first paragraph has $f(v_0) = g(v_0) + g'(v_0) - 1$, and that even the best choice of lists on G can only knock out at most $g(v_0) - 1$ "slots" on v_0, leaving $g'(v_0)$ slots to color G' with, which is enough. However, if you drop $f(v_0)$ down by any more, then this breaks down, and a clever choice of lists on G, whose existence is guaranteed by the previous lemma, can knock out $g(v_0) - 1$ slots, leaving less than $g'(v_0)$ slots on v_0, and so the remaining list sizes on G' sum up to less than the sum choice number of G'.

BBBD's Theorem 3 can be rewritten using the τ, ρ terminology in the form of the following theorem.

Theorem 4 (BBBD Theorem 3)**.** *For any graph G, $\chi_{\mathrm{SC}}(G) \leq \rho(G)$ with equality if and only if there exists a simple minimum choice function.*

Proof. The inequality $\chi_{\mathrm{SC}}(G) \leq \rho(G)$ follows immediately from Lemma 2.2 of the thesis. Next, if f is a minimum choice function with $f(v) = 1$ (resp. $\deg(v) + 1$), then $G - v$ is f^v-choosable (resp. f_{G-v}-choosable). Thus, $\chi_{\mathrm{SC}}(G - v) \leq \chi_{\mathrm{SC}}(G) - \deg(v) - 1 = \mathrm{size}(f^v) = \mathrm{size}(f_{G-v})$. Rearranging this yields $\chi_{\mathrm{SC}}(G) \geq \rho(G)$. For the converse, since

$\chi_{\mathrm{SC}}(G) = \rho(G)$, there exists a vertex v such that $\chi_{\mathrm{SC}}(G) = \chi_{\mathrm{SC}}(G - v) + \deg(v) + 1$. Let f be a minimum choice function on $G - v$. Define a size function g on G by $g(w) = f(w)$ for $w \neq v$, and $g(v) = \deg(v) + 1$. Then as $g(v) = \deg(v) + 1$, it is simple, and as $\operatorname{size} g = \chi_{\mathrm{SC}}(G - v) + \deg v + 1 = \chi_{\mathrm{SC}}(G)$, G is g-choosable. $\square$

Alternatively, at the last step in the proof, we could have reached the desired conclusion by instead defining $g(w) = f(w) + 1$, if v is adjacent to w, $g(v) = 1$, and $g(w) = f(w)$ for any other vertex w. Notice that $g^v = f$.

Edge counterexample — BBBD asked if it were true that, given a minimum choice function f, there exists an f-assignment forcing an edge. The answer is no, as the fan graph F_5 with the size function assigning list size 4 to the fan vertex v_0, and $(2, 2, 2, 3, 2)$ to the path, is choosable, but the edge v_0v_1 can't be forced.

Of more interest than the sum choice number, perhaps, is $\gamma_{\mathrm{SC}}(G) = GB(G) - \chi_{\mathrm{SC}}(G)$, the gap between the greedy bound and sum choice number.

Lemma 5. *If G is connected, then for any $v \in V(G)$, $\chi_{\mathrm{SC}}(G) \geq \chi_{\mathrm{SC}}(G - v) + 2$.*

Proof. If G were f-choosable for an f of size $\chi_{\mathrm{SC}}(G - v) + 1$, then $f(v) = 1$, as otherwise, $\operatorname{size}(f_{G-v}) < \chi_{\mathrm{SC}}(G - v)$. Since G is connected, v has a neighbor in G, so $\operatorname{size}(f^v) < \chi_{\mathrm{SC}}(G - v)$, and therefore $G - v$ is not f^v-choosable. This is not possible, as Lemma 2.1 (of the thesis) says that G is f-choosable if and only if $G - v$ is f^v-choosable. $\square$

Lemma 6. *If $G = (V, E)$ is connected, then $\gamma_{\mathrm{SC}}(G) \leq \min_{v \in V} \gamma_{\mathrm{SC}}(G - v) + \deg(v) - 1$. In particular, if $G - v$ is sc-greedy for all v, then $\gamma_{\mathrm{SC}}(G) \leq \delta(G) - 1$.*

Proof. Let $v \in V(G)$. Since G is connected, $\chi_{\mathrm{SC}}(G) \geq \chi_{\mathrm{SC}}(G - v) + 2$. Hence $|V| + |E| - \chi_{\mathrm{SC}}(G) \leq |V| + |E| - \chi_{\mathrm{SC}}(G - v) - 2$. The result follows immediately from the relations $|V(G - v)| = |V| - 1$ and $|E(G - v)| = |E| - \deg(v)$. $\square$

For a vertex v of minimum degree, the preceding lemma implies that $\gamma_{\mathrm{SC}}(G) - \gamma_{\mathrm{SC}}(G - v) \leq \delta(G) - 1$.

Lemma 7. *If H is an induced subgraph of G, then $\gamma_{\mathrm{SC}}(G) \geq \gamma_{\mathrm{SC}}(H)$.*

Proof. There exists a choice function f on H of size $\chi_{\mathrm{SC}}(H)$. Next, choose any ordering of the vertices of G such that no vertex of $G - H$ comes before a vertex of H in the ordering, and let g be a size function on G defined by greedy coloring on this ordering, $g(v_i) = 1 + |\{v_j : i < j, \text{ and } v_iv_j \in E(G)\}|$. Let h be the size function on G defined by $h_H = f$ and $h_{G-H} = g$. It is easy to see that G is h-choosable, and $\operatorname{size} h = \chi_{\mathrm{SC}}(H) + \mathrm{GB}\,(G - H)$. We may thus conclude that $\chi_{\mathrm{SC}}(G) \leq \chi_{\mathrm{SC}}(H) + \mathrm{GB}\,(G - H)$ and $\gamma_{\mathrm{SC}}(G) \geq \gamma_{\mathrm{SC}}(H)$. $\square$

Here are my versions of the statements of Lemmas 7 and 8 of Isaac's Sum List Coloring Block Graphs paper. The proofs there are straightforward. Note that in (b), $t_i \geq i$ for all i is equivalent to saying that H is f_H-choosable.

Lemma 8. *Let (G, f) be given, and let H be an induced subgraph of G such that any two vertices of H have same neighborhood N outside of H.*

(a) Let g be a size function on H, given by $g(v) = f(v) - |N|$. If H is g-choosable, then G is f-choosable if and only if $G - H$ is f_{G-H}-choosable.

(b) Suppose H is a k-clique. Let $t_1, \dots t_k$ be the list sizes of the vertices of H ordered such that $t_1 \leq t_2 \leq \cdots \leq t_k \leq k$. If $t_i \geq i$ for all i, then G is f-choosable if and only if $G - B$ is f^H_{G-H}-choosable.

Fact 1. $2 \times n$ ***array*** — *The sum choice number of the $2 \times n$ array, $K_2 \square K_n$, as calculated by Isaac, is $n^2 + \lceil 5n/3 \rceil$. Since the greedy bound is $n^2 + 2n$, this means that $\gamma_{\mathrm{SC}}(K_2 \square K_n) = \lfloor n/3 \rfloor$. This means that the wrap-around ladder, $P_2 \square C_3$, is not sc-greedy.*

Here is Isaac's Lemma 1 from the $2 \times n$ array paper. This may be useful.

Lemma 9. *Let f be a choice function on K_n with size $\chi_{\mathrm{SC}}(K_n) + t$. The f-assignment consisting of initial lists forces at least $n - 2t$ vertices.*

The proof is not hard, just put the vertices in order of increasing list size, and whenever $f(v_{i-1}) = i - 1$ and $f(v_i) = i$ a color is forced on v_i. There are at most $2t$ indices at which $f(i) \neq i$, so at most $2t$ vertices can't be forced. Is it possible to force any more than this? Of course, use list sizes 1,2,3,6, so that $t = 2$, $n - 2t = 0$, but 3 vertices are forced.

Question 1. *Is there some nice way of showing that G is f choosable, where* size $f <$ *$\chi_{\mathrm{SC}}(G)$? In my $P_3 \square P_n$ proof, and Isaac's $K_2 \square K_n$ most of the work was in showing that a certain small graph ($P_3 \square P_3$ for mine, and $K_2 \square K_3$) was choosable.*

Theorem 4 in BBBD is the following.

Theorem 10. *The graph obtained from K_n by attaching a vertex to each of k different vertices of K_n is sc-greedy.*

I think it actually follows relatively easily from the previous lemma, though I don't yet have the proof worked out. From this they prove the following statement.

Theorem 11. *Let T be a tree on $n \geq 3$ vertices. There exists a sequence of connected graphs G_i, each sc-greedy, where $G_1 = K_n$, $G_n = T$ and G_{i+1} is obtained from G_i by deleting an edge.*

The proof is an easy induction.

Observation 1. *BBBD also introduce the notion of* sc-critical *— a graph G is sc-critical provided that $\chi_{\mathrm{SC}}(G-e) < \chi_{\mathrm{SC}}(G)$ for every edge e of G. Paths, cycles, and complete graphs are sc-critical, but in general, removing an edge from an sc-greedy graph might not produce another sc-greedy graph, so not all sc-greedy graphs are sc-critical. Examples of graphs which are not sc-critical are lots of the complete bipartite graphs, though not all.*

Fact 2. *Here is a table of all the graphs I know of that have $\gamma_{\mathrm{SC}} > 0$.*

G	$\gamma_{\mathrm{SC}}(G)$
$\theta_{1,1,2k+1}$	*1*
$K_1 \vee P_n$	$\lfloor (n+1)/11 \rfloor$
$K_2 \,\square\, K_n$	$\lfloor n/3 \rfloor$
$P_3 \,\square\, P_n$	$\lfloor n/3 \rfloor$
$K_{2,n}$	$n - \lfloor \sqrt{4n+1} \rfloor + 1$
$K_{3,n}$	$2n - \lfloor \sqrt{12n+4} \rfloor + 2$

Question 2. *Theta graphs of the form $\theta_{1,1,2n+1}$ are interesting in that they are the only example of have of a sequence of graphs, which are inductively defined, that is $\theta_{1,1,2n+1}$ is obtained from $\theta_{1,1,2n-1}$ by attaching vertices (and it's the same type of attaching regardless of k), that have $\gamma_{\mathrm{SC}} = C > 0$, where C is a constant (independent of n). All of the other inductively defined sequences of graphs fall under two categories: P_n, C_n, K_n, … all have $\gamma_{\mathrm{SC}} = 0$, or $K_{m,n}$, $K_2 \,\square\, K_n$, $P_3 \,\square\, P_n$, $K_1 \vee P_n$ all have γ_{SC} a function of n. What other inductive classes of graphs have γ_{SC} independent of m?*

Question 3. *Is the following true in general, or, at least, in certain useful special cases? If v and w are adjacent and $f(v) \le f(w)$, then it is only necessary to consider the case where $\mathcal{C}(v) \subset \mathcal{C}(w)$? More precisely, is the following statement true (or true in some sense)? If $f(v) \le f(w)$ and G is not f-choosable, then there exists an f-assignment $\mathcal{C}$ that has no proper coloring and satisfies $\mathcal{C}(v) \subset \mathcal{C}(w)$.*

Question 4. *Does $K_{p,q}$ maximize γ_{SC}?*

Question 5. *I think this is true: If $\gamma_{\mathrm{SC}}(G) > \gamma_{\mathrm{SC}}(G-v)$ for all v in $V(G)$, then there does not exist a simple minimum choice function. An analogous result, maybe with some sort of condition with how things fit together, should hold for 1-configurations. Is this an iff? Also, the similar earlier result about if $\gamma_{\mathrm{SC}} = \rho$ is that just obvious? Does it need a proof?*