Creative Approaches to Discrete Mathematics

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Reasoning question

Consider the following statement:

If G is Keplerian, then $|G|$ is even.

Assuming that "Keplerian" actually means something, answer the following:

1. Suppose $|G|$ is even. What can you conclude about whether G is Keplerian or not?
2. Suppose G is not Keplerian. What can you conclude about $|G|$?
3. Is $|G|$ being even a necessary, sufficient, or both necessary and sufficient condition for G to be Keplerian?

A notation/mathematical language problem

The following is an easy problem that I am purposely making look difficult:

> *Let $m \in \mathbb{N}$. Let $d(m,1)$, $d(m,2)$, $\ldots d(m,n)$ denote all the positive integers x for which there exists a $y \in \mathbb{Z}$ such that $xy = m$. Let $\sigma(m) = \sum_{i=1}^{n} d(m,i)$. An integer m is called teleiotic if $\sigma(m) = 2m$. Prove that prime numbers are not teleiotic.*

- A traveler has to get a wolf, a goat, and a cabbage across a river. The wolf can’t be left alone with the goat, the goat cant be left alone with the cabbage, and the boat can only hold the traveler and a single animal/cabbage at once.

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- Water pouring problems can be similarly represented.

Graphs — Variant of Kevin Bacon problem

- Given two actors, A and A', find the shortest path between them such that A was in a movie with B who was in a movie with C .... who was in a movie with A'.

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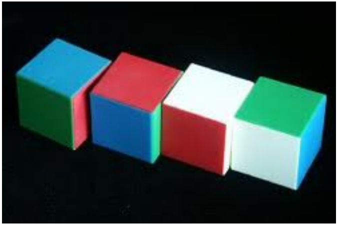
- *Solution:*
  - vertices are movies and actors
  - edge between two vertices iff one is a movie and the other is an actor in that movie
  - run a breadth-first search
- Can get a big data file online and have some fun

- Have to stack the blocks so that no color is repeated on any of the four sides.



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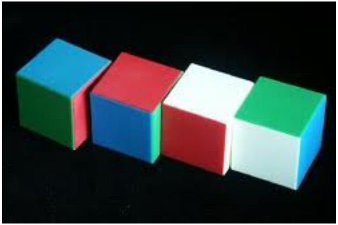

- *Solution*
  - Vertex for each color
  - Edge if two colors are on opposite sides of the cube
  - Label edge with the color name
  - Find two disjoint Hamiltonian cycles that include edges labeled with each color name

- Table of distances between radio stations:

A	B	C	D	E	F	G
A	55	110	108	60	150	88
B		87	142	133	98	139
C			77	91	85	93
D				75	114	82
E					107	41
F						123

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- Lots of similar problems to this can be made...

- Below is a list of the exams (letters) each student (numbers) took. Is it possible to schedule the exams on three separate days so no student would be scheduled for two exams at once?

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Middle levels problem

List the 2- and 3-element subsets of $\{1, 2, 3, 4, 5\}$ according to the following rules:

- Start and end with 12. Each other subset appears exactly once.
- Alternate between 2- and 3-element subsets, such that each subset in the list is obtained from the previous one by either adding or deleting a single element.

A possible answer:

$$12, 123, 13, 134, 34, 345, 35, 135, 15, 25, \ldots, 125, 12$$

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  - 1 is a losing position.
  - Therefore 2 and 3 are winning positions.
  - Therefore 4 is a losing position.
  - Therefore 5 and 6 are winning positions...
  - Losing positions: 1, 4, 7, 10, 13, ...

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  Idea of solution:
  $123 \rightarrow 4123,\ 1423,\ 1243,\ 1234$
  $132 \rightarrow 4132,\ 1432,\ 1342,\ 1324$
  $\ldots$

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- $x^2 + y^2 = z^{2n}$ has an integer solution for all $n \geq 1$.

  Idea of solution: Multiply $x^2 + y^2 = z^2$ through by z^2 to get $(xz)^2 + (yz)^2 = z^4$

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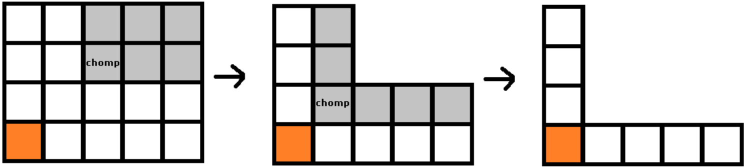
- Every tree can be colored with two colors.

  Idea of solution: Pull off a leaf. Left with a tree on less vertices and can easily color the leaf.

Chomp: The bottom left square is poisoned. Players take turns chomping parts of the candy bar off. When a player chomps off a portion, they pick a square and everything above and to the right of that square is chomped off, like in the figure below. The player who is stuck at the end with the poisoned square loses.



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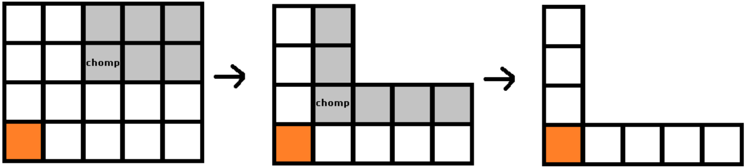
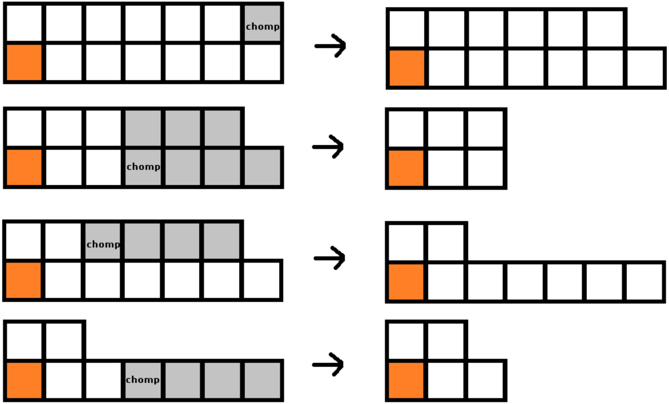


What is the ideal strategy for the $2 \times n$ case?

Non-algebraic induction problems – Chomp solution

Idea of solution: A $2 \times k$ with the top right square missing is a losing position. All others are winning positions.



Function problem

$A = \{a, b, c\}$.

$A* = a, b, c, aa, ab, ac, ba, \ldots$ be all words with letters from A.

$f : A^* \to A^*$ by replacing every b in w with a.

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$A* = a, b, c, aa, ab, ac, ba, \ldots$ be all words with letters from A.

$f : A^* \to A^*$ by replacing every b in w with a.

1. Find the range of f.
2. One-to-one? Onto?
3. Find $f^{-1}(\{aa\})$.

Function problem

$f(G)$ is number of odd-degree vertices in graph G.

1. Find $f(P_5)$ where P_5 is a path on five vertices.
2. One-to-one? Onto?
3. What is $f^{-1}(\{4\}) \cap \{G \in \mathcal{G} : G \text{ has an Eulerian tour}\}$?

Function problem

Let $A = \{1, 2, 3, 4, 5\}$. $f : \mathcal{P}(A) \to \mathbb{Z}$, defined by $f(X) = |X| \bmod 2$.

1. Range?
2. $f(B)$, where $B = \{\{1, 2\}, \{1, 2, 3\}, \{2, 3, 4\}\}$.
3. $f^{-1}(\{1\})$
4. One-to-one? Onto?

Function problem

- The Caesar shift cipher shifts every character by 3 in the alphabet, wrapping around as necessary.

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- The Caesar shift cipher shifts every character by 3 in the alphabet, wrapping around as necessary.
- $f : \{a, b, \ldots z\} \to \mathbb{Z}$ maps each letter to its position in the alphabet
- Find a formula for it in terms of f and a shifting function

Find the equation of a function. . .

- On a certain 50-question test you get a point for each correct answer and lose a point for every four incorrect answers. Find a function, $s(x)$, giving the score on the test, where x is the number of correct answers.

- Let x be a measurement in inches. We want functions f and g that convert x to a measurement in feet and inches. For instance, 74 inches is 6 feet, 2 inches, and we would want $f(74) = 6$ and $g(74) = 2$.

Find the equation of a function...

- Suppose you put \$4000 in the bank and it collects interest at a rate of 3%, compounded yearly. No further money is added to the account nor is any money ever withdrawn. Create a function $f(x)$ that gives the number of years it takes for the account to be worth x dollars.

- In 2007, the postage rate for a certain type of mail up to 13 ounces is 25 cents for the first ounce or fraction thereof and 17 cents for each additional ounce or fraction thereof. Create a function $P(w)$ giving the postage rate for w ounces.

Find the equation of a function...

- Two ways of describing the entries in a table. Find a function f that maps the first scheme into the second scheme. For instance $f(0,0) = 0$, $f(1,1) = 5$, and $f(3,2) = 14$.

(0,0)	**0**	(0,1)	**1**	(0,2)	**2**	(0,3)	**3**
(1,0)	**4**	(1,1)	**5**	(1,2)	**6**	(1,3)	**7**
(2,0)	**8**	(2,1)	**9**	(2,2)	**10**	(2,3)	**11**
(3,0)	**12**	(3,1)	**13**	(3,2)	**14**	(3,3)	**15**

- Come up with a formula for f^{-1} in the problem above.

First writing assignment

Prompt: The following problems on homework 2 gave a lot of students a lot of trouble:

2. For each question below, indicate whether the blank should be filled by *necessary*, *sufficient*, *both necessary and sufficient*, or *neither necessary nor sufficient*

   (a) All multiples of 9 are divisible by 3. Being divisible by 3 is a ______ condition for being a multiple of 9.

   (b) If $\{s_n\}$ is Cauchy-summable, then $\prod s_n < 1$. The condition $\prod s_n >= 1$ is a ______ condition for $\{s_n\}$ *not* being Cauchy-summable.

   (c) If X is Riemannian, then it is also Gaussian. X being Riemannian is a ______ condition for X being Gaussian.

3. Consider the following statement.

   If $x + y < 10$, then $f(x, y) = 0$.

   (a) If $x + y = 15$, what can we conclude from the statement about $f(x, y)$?

   (b) If $x + y = 5$, what can we conclude from the statement about $f(x, y)$?

   (c) If $f(x, y) = 1$, what can we conclude from the statement about $x + y$?

   (d) If $f(x, y) = 0$, what can we conclude from the statement about $x + y$?

These problems cover the concepts of necessary/sufficient and the concepts of converse, contrapositive, and what we can and can't logically conclude from statements. Imagine you have to explain these concepts and how to do the problems to someone in class who is having trouble. In a short paper of 300-600 words, do just that. You must do more than just restate the answers. Take some time to clearly explain things.

Second writing assignment

Prompt: The following problems on homework 4 gave a lot of students a lot of trouble:

2. Let $A = \{a, b, c\}$ and define $f : A^* \to \mathbb{Z}$ such that $f(w)$ is the number of a's in w. Determine the following:
   (a) The domain of f
   (b) The codomain of f
   (c) The range of f
   (d) $f^{-1}(\{1, 2\})$
   (e) Is f onto? Why or why not?
   (f) Is f one-to-one? Why or why not?
   (g) Is f a bijection? Why or why not?
   (h) Does f have an inverse? Why or why not?

3. A prime number is a number greater than 1 whose only divisors are itself and 1. Let $f : \{3, 4, 5 \dots\} \to \mathbb{N}$ be a function defined such that $f(n)$ is the largest prime number less than n.
   (a) Is f one-to-one?
   (b) Is f onto?

Just like in the last paper, explain the concepts necessary to do the above problems as if you were explaining them to someone in class who is having trouble. In doing so, answer some or all of the parts of these problems. You must do more than just restate the answers. Take some time to clearly explain things.

Third writing assignment

1. Prove or disprove the following statements.
   (a) For all real numbers a and b, if $a < b$, then $a^2 < b^2$.
   (b) The product of two odd integers is odd.
   (c) $\lfloor -x \rfloor = -\lceil x \rceil$
   (d) If $f : \mathbb{R} \to \mathbb{R}$ is a one-to-one function, then $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$.
   (e) No primes other than 2 and 3 leave a remainder of 0, 2, 3, or 4 when divided by 6.
   (f) If $n \in \mathbb{N}$, then $n^2 + n + 41$ is prime.
   (g) None of the integers in the following sequence are prime:

   $$9, 98, 987, 9876, 98765, 9876543, 98765432, 987654321, 9876543219, 98765432198, \ldots$$

   Hint: [A number is divisible by 3 if and only if the sum of its digits is divisible by 3.]
2. On an island there are two tribes. One tribe's members always tell the truth, while the other tribe's members always lie. You meet three tribesmen. The first one says "All of us are liars." The second then says, 'Only two of us are liars." Finally, the third one says "Only one of us is a liar." Give a complete and logical explanation (without using truth tables) of who belongs to what tribe and why.
3. Prove, without using calculus, that if $ab = 36$, then $a + b$ is minimized when $a = b = 6$.

The class as a whole had an average of 24/45 on that homework. This writing assignment is an opportunity for you to better learn some of those proofs as well as help out your grade on that assignment. This writing assignment will be graded on its own, just like the other writing assignments. If your score on this assignment is better than your score on the homework, then you can also replace your homework score with your score on this assignment (scaled out of 45 points).

Fourth writing assignment

Purpose

The purpose of this assignment is to help you better understand mathematical induction.

Prompt

Weave the following together into a coherent introduction to induction.

1. Explain idea of induction and why it works.
2. Then give some general strategies for how to do induction proofs.
3. Do at least two example problems in detail.

Audience

Your audience are other students in Discrete Math who are having difficulty with induction.

Final project

8-page multidraft paper

- ~~How random numbers are generated~~
- ~~Cryptography~~
- ~~Bitwise operations~~
- Ackermann's function and really large numbers
- ~~Four color theorem~~
- ~~Information theory~~
- ~~Bioinformatics~~
- P=NP problem
- Cellular automata
- ~~Game theory~~
- ~~Voting theory~~
- Graph algorithms
- Ramsey theory
- ~~Pythagorean triples~~
- ~~Mathematics of Rubik's Cubes~~
- ~~Hashing~~
- ~~Magic squares~~
- Number bases and number systems
- ~~Error-correcting codes~~
- Combinatorial designs
- ~~Logical paradoxes~~
- Irrational and transcendental numbers
- Interesting primality tests
- ~~Mathematical card and magic tricks~~
- Recursive algorithms
- ~~Ant colony algorithms~~
- ~~Monte Carlo simulations~~
- Regular expressions
- L-systems
- ~~Solving recurrence relations~~
- ~~Bayes' theorem and consequences~~
- Analyzing social networks and large data sets
- Mathematics of origami