Patterns and Number Theory

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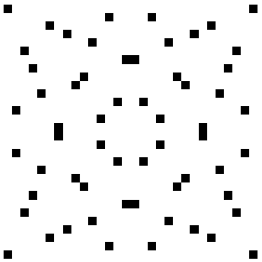
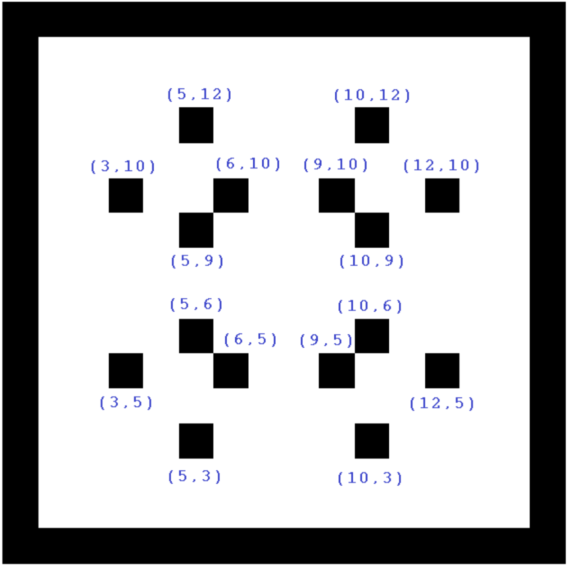


This is work with Jackie Kearney who researched this for her senior honors project.

What we did

- Plot $\{(x, y) : f(x, y) \equiv 0 \pmod{n}\}$
- Various functions $f(x, y)$ and values of n
- Usually x, y between 0 and n or $2n$

$\{(x,y) : xy \equiv 0 \pmod{15}\}$



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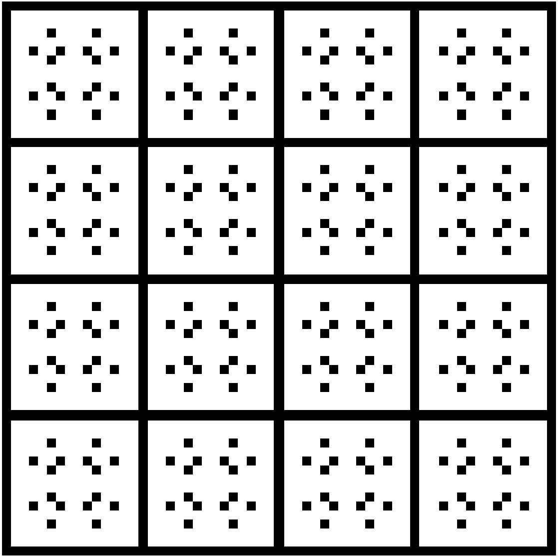

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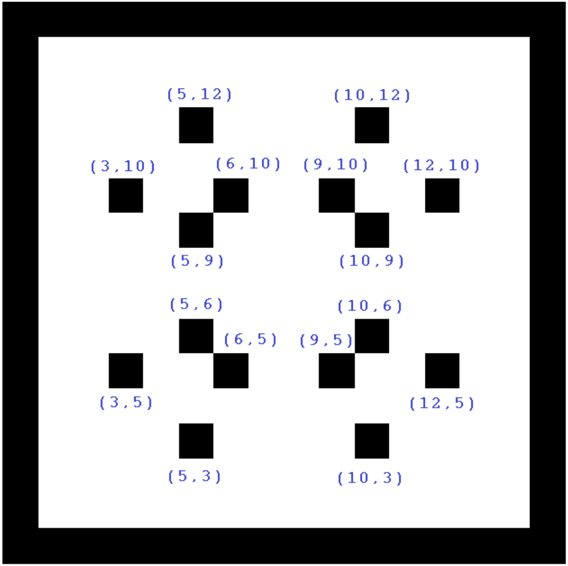
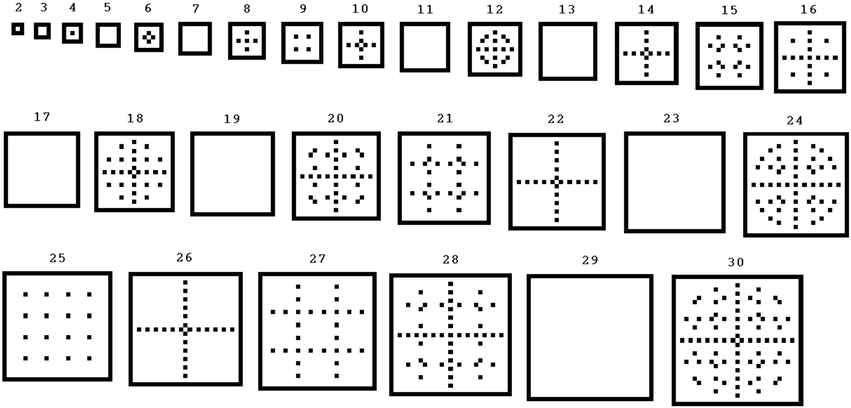


Symmetry

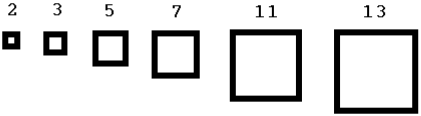
- If $xy \equiv 0 \pmod{n}$, then $(n-x)y \equiv ny - xy \equiv 0 \pmod{n}$.
- Similarly $x(n-y) \equiv 0 \pmod{n}$
- As x and y are interchangeable, there is symmetry across $y = x$

$\{(x,y) : xy \equiv 0 \pmod{n}\}$ for $n = 1$ to 30



Blank boxes

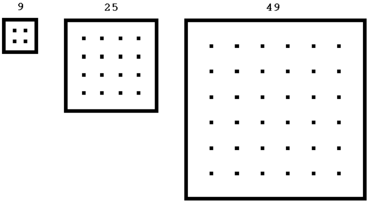
Get a blank box if n is prime



$xy \equiv 0 \pmod{n}$
$\Leftrightarrow n \mid xy$
Euclid's Lemma $\Rightarrow n \mid x$ or $n \mid y$
But $0 < x, y < n$.

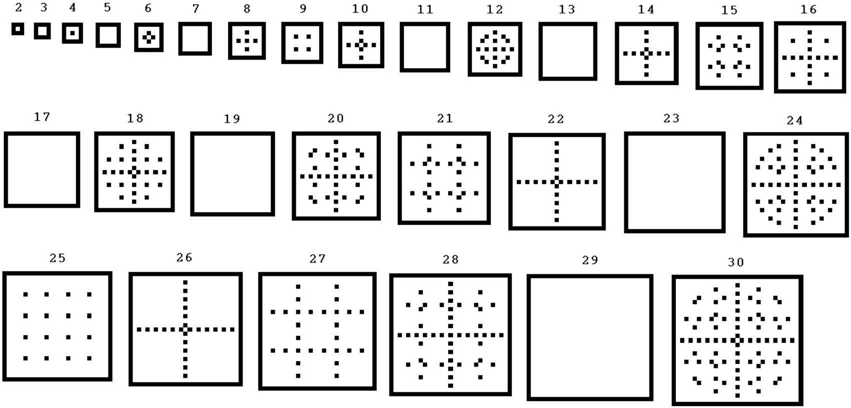
Grids

Get a grid pattern if $n = p^2$ for an odd prime p.

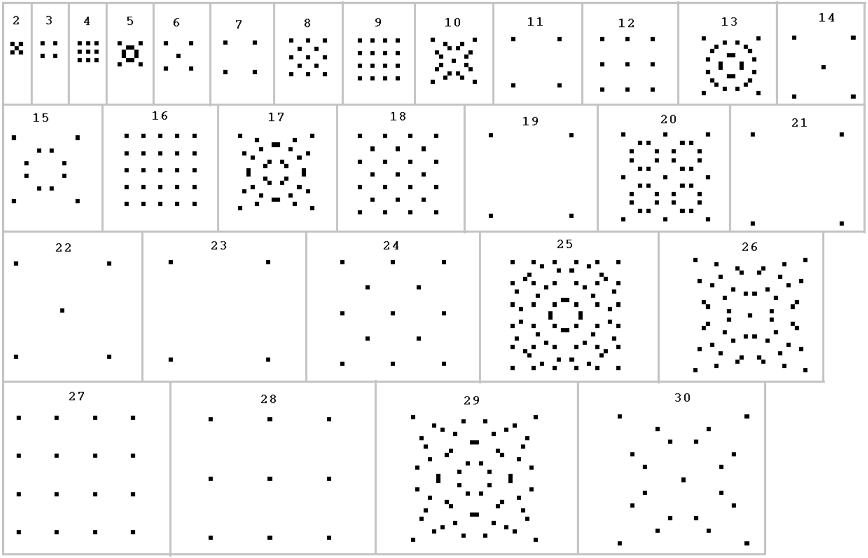


$xy \equiv 0 \pmod{p^2}$
$\Leftrightarrow p^2 \mid xy$
Then $p \mid xy$.
Euclid's Lemma implies $p \mid x$ or $p \mid y$.
So only get points of form (ip, jp)

$\{(x,y) : xy \equiv 0 \pmod{n}\}$ for $n = 2$ to 30



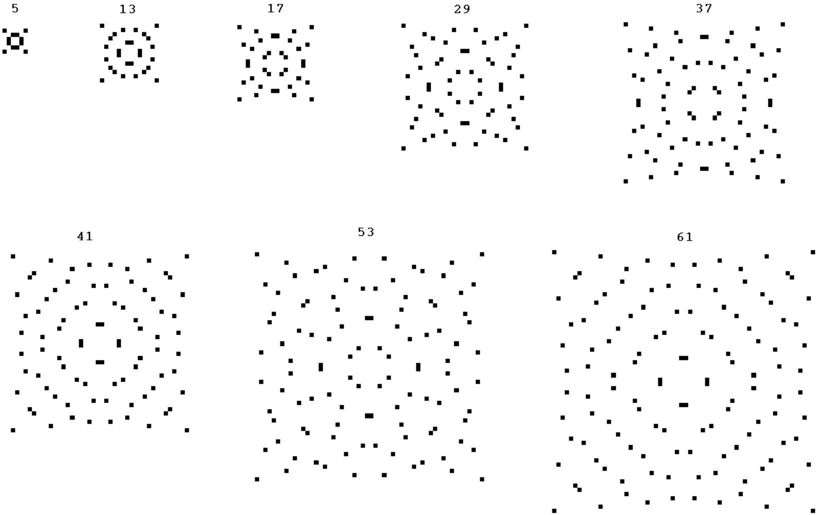
$\{(x, y) : x^2 + y^2 \equiv 0 \pmod{n}\}$ for $n = 2$ to 30



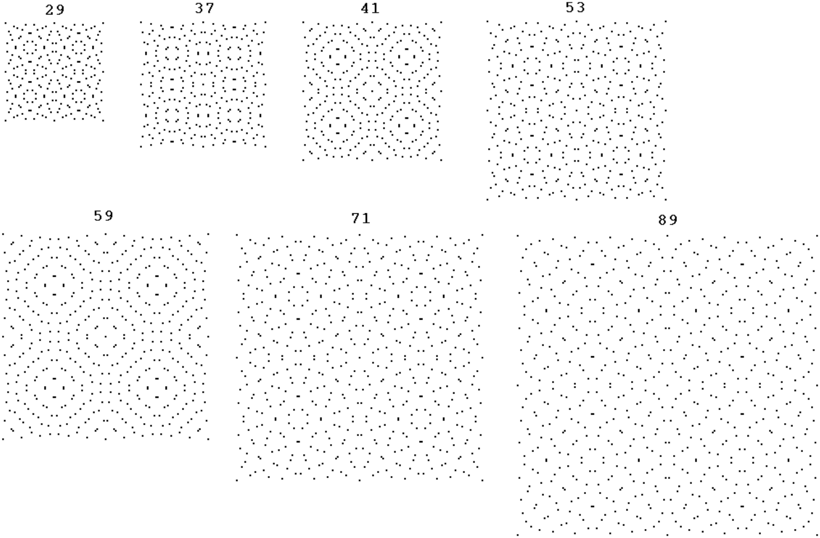
$4k + 1$ vs $4k + 3$

- Theorem of Fermat: An odd prime is the sum of two squares if and only if it is of the form $4k + 1$.

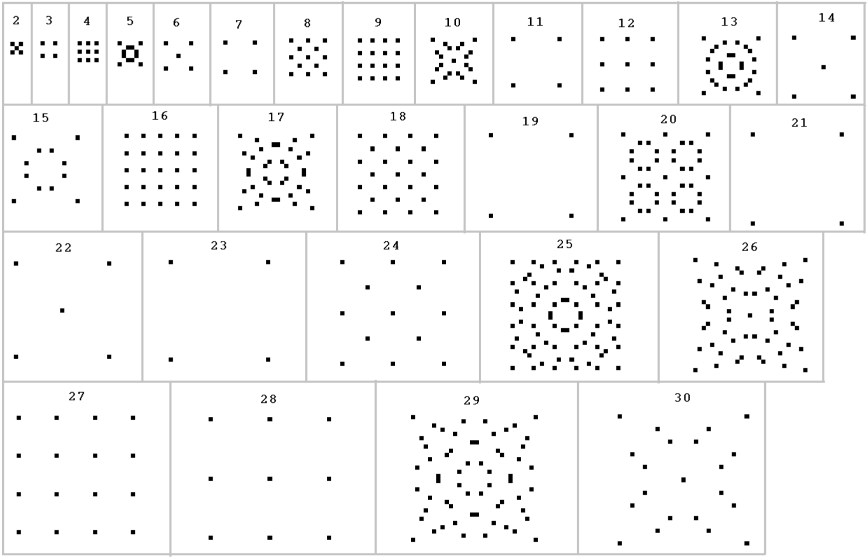
$4k + 1$ primes



$4k + 1$ primes (plots in range 0 to $2n$)



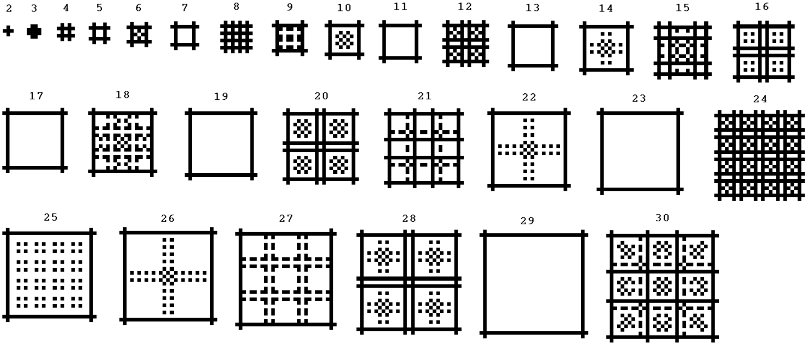
$\{(x, y) : x^2 + y^2 \equiv 0 \pmod{n}\}$ for $n = 2$ to 30



Other patterns

- n is the sum of two squares iff each $4k+3$ prime in the prime factorization of n is raised to an even power.
- This explains:
  - Why 21 is also blank
  - Why various types of grids appear

$\{(x,y) : (x^2-1)(y^2-1) \equiv 0 \pmod{n}\}$



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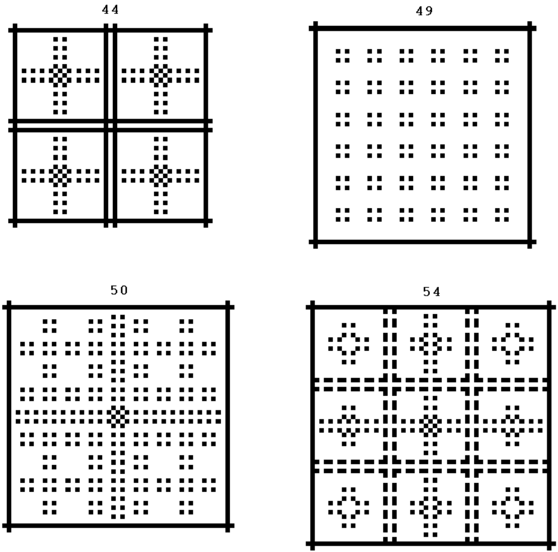

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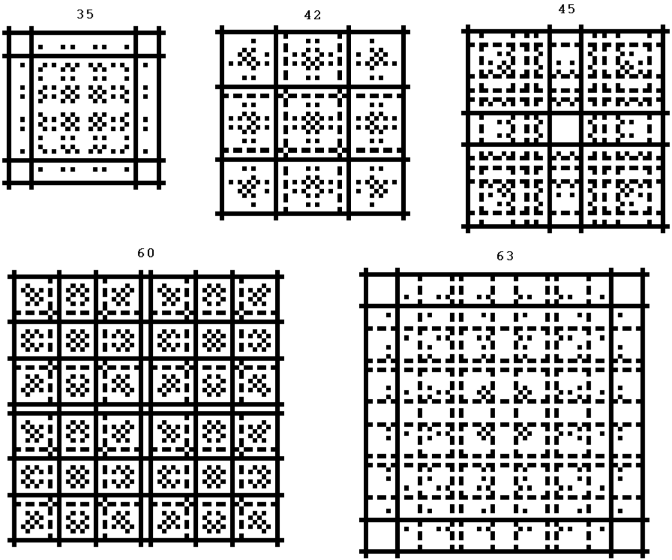
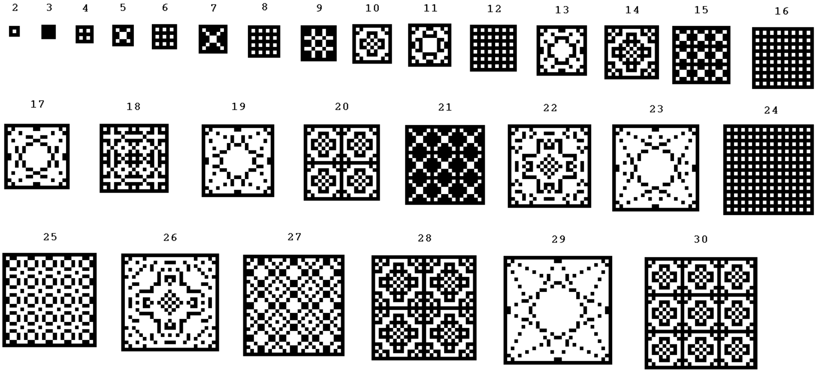


$\{(x, y) : xy(x^2 - 4y^2)(4x^2 - y^2) \equiv 0 \pmod{n}\}$



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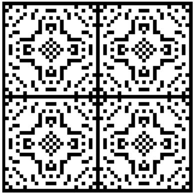

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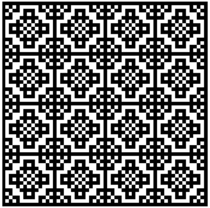

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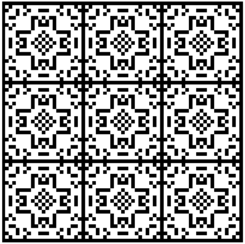

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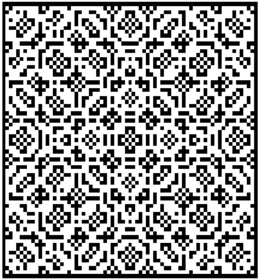
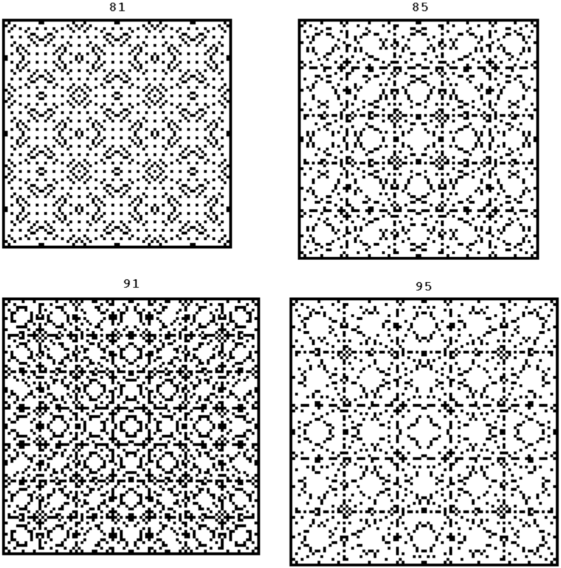
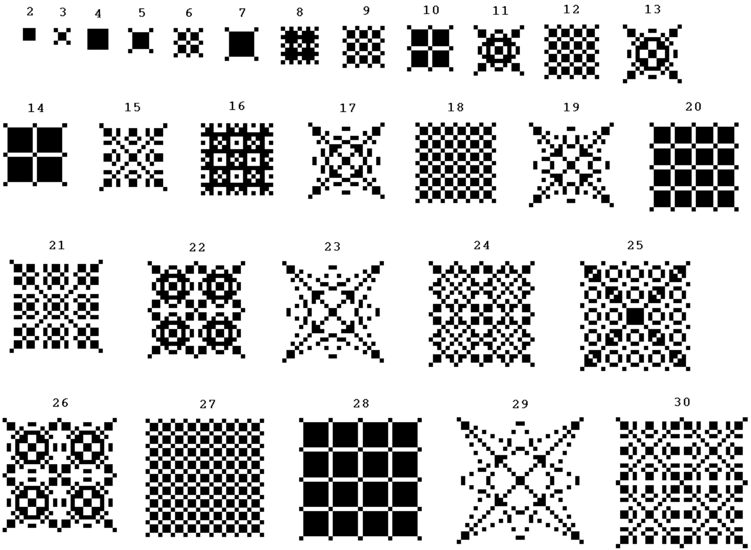


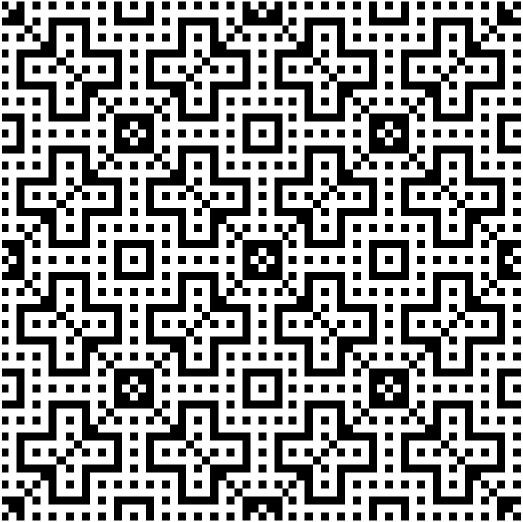
$\{(x,y) : xy(x^2-4y^2)(4x^2-y^2) \equiv 0 \pmod{n}\}$



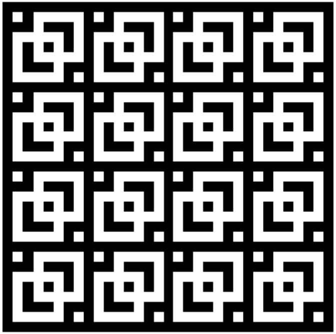
$$\{(x,y) : (x^2 - y^2)(x^2 - 4y^2)(4x^2 - y^2) \equiv 0 \pmod{n}\}$$



$\{(x,y) : (x^2 - y^2)(x^2 - 4y^2)(4x^2 - y^2) \equiv 0 \pmod{64}\}$



$\{(x, y) : xy(x^4 - y^4) \equiv 0 \pmod{n}\}$



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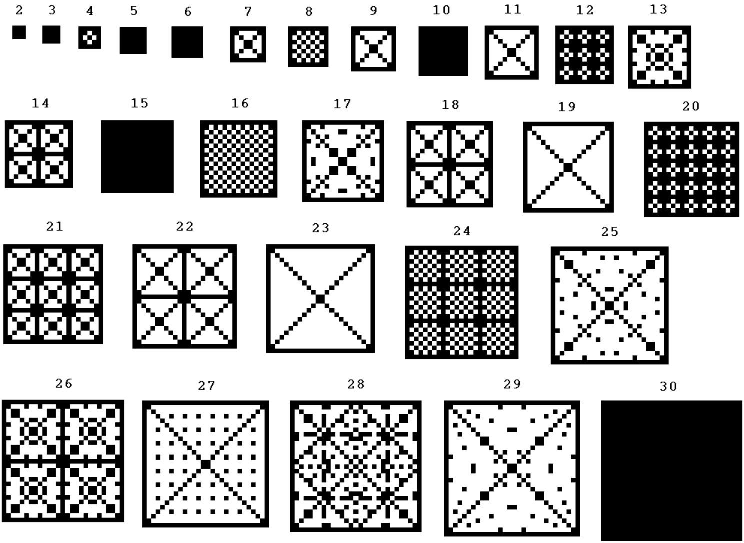

Filled box for 30

- $xy(x^4 - y^4) = xy(x-y)(x+y)(x^2+y^2)$
- $30 = 2 \cdot 3 \cdot 5$
- $x = 3k + r$, $y = 3k' + r'$, $r, r' \in \{0, 1, 2\}$.
- If $r = r'$, then $3 \mid (x - y)$.
- If $r \neq r'$, then $3 \mid (x + y)$.
- $x = 5k + r$, $y = 5k' + r'$, $r, r' \in \{0, 1, 2, 3, 4\}$.
- If $r = r'$, then $5 \mid (x - y)$.
- If $(r, r') = (1, 4)$ or $(2, 3)$, then $5 \mid (x + y)$
- If $(r, r') = (1, 3)$, $(2, 4)$, then $5 \mid (x^2 + y^2)$.

$x^2 + y^2 \bmod 12$