Some unusual mathematical images and the math behind them

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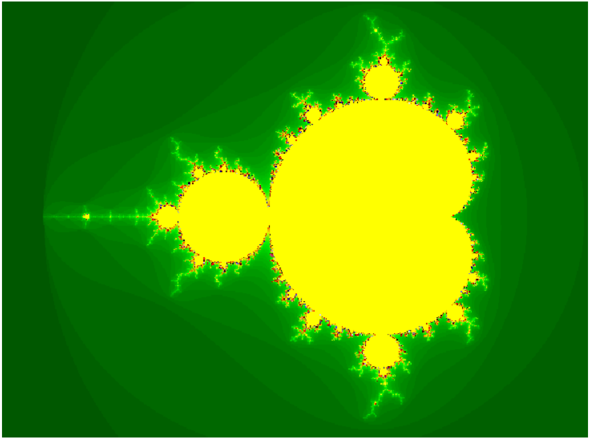
A little history



- Circa 1999 wanted to draw Mandelbrot set
- Had some programming experience
- What about other formulas?

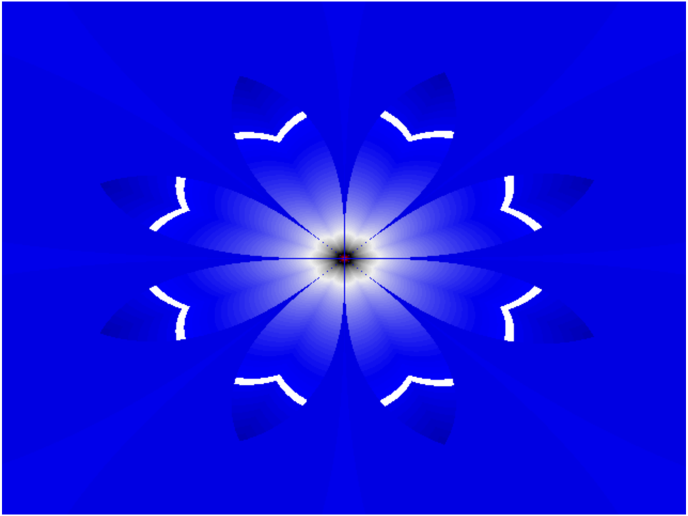
```
pf[a[0]].left = LEFT_END;
if (pf[1].fa_loc == -1)  // special case if there is only one item in the formula
  pf[a[0]].right = RIGHT_END;
else
  {
  for (i = 0; pf[i+1].fa_loc != -1; i++)
    {
    if (form_arr[i].pri == FUNCTION && func[form_arr[i].buffer_loc].num_arg > 1)
      {
      /* scan right, if we fall off the edge of the formula or if we reach the end
         of the function call without finding a comma at the next paren_level up,
         then return an error, otherwise set the right field accordingly */

      if (form_arr[i+1].pri != PARENTHESIS)
        return 9;

      for (j = i+2; pf[j].fa_loc != -1 &&
           form_arr[j].paren_level > form_arr[i].paren_level &&
           !(form_arr[j].pri == COMMA && form_arr[j].paren_level == form_arr[i].pare

      if (form_arr[j].paren_level > form_arr[i].paren_level)
        {
        for (k = j-1; k>0 && form_arr[k].pri == PARENTHESIS; k--) {}
        pf[a[i]].right = a[(pf[k].fa_loc == -1) ? j-1 : k];
        }
      else
        return 9;
      }
```

The first image I generated



Complex numbers

$i = \sqrt{-1}$ (solution to $x^2 + 1 = 0$)

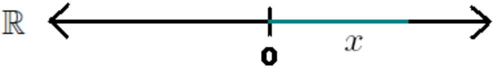
Examples: $2i, \quad 3 + 4i, \quad -.2 + .76i$

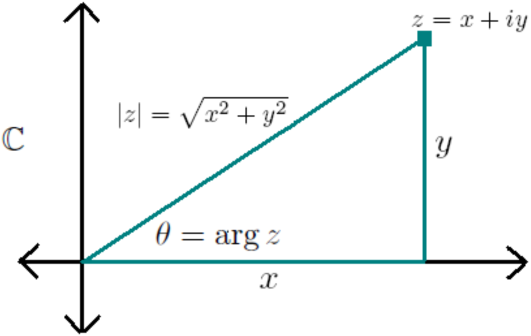
Addition: $(2 + 3i) + (5 + 8i) = 7 + 11i$

Multiplication: $(2 + 3i)(5 + 8i) = 10 + 31i + 24i^2 = -14 + 31i$

Division: $\frac{2+3i}{5+8i} = \frac{2+3i}{5+8i} \cdot \frac{5-8i}{5-8i} = \frac{34-8i}{89} = \frac{34}{89} + \frac{8}{89}i$

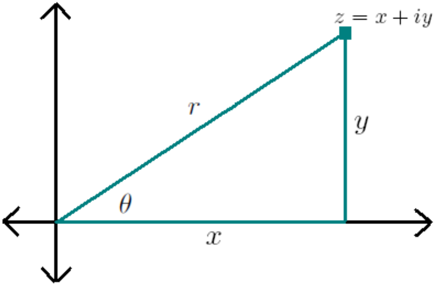
Picturing them





Polar representation

$x + iy \longleftrightarrow re^{i\theta}$



$z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}$

$\Rightarrow z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$ (a rotation and a dilation)

Iteration

Example: Let $f(x) = x^2$ and start with $x = 2$.

$f(2) = 4$
$f(4) = 16$
$f(16) = 256$
$f(256) = 65536$
$\dots$

Iterates are approaching ∞.

A different starting point

Let $f(x) = x^2$ and start with $x = \frac{1}{2}$.

$f(\frac{1}{2}) = \frac{1}{4}$

$f(\frac{1}{4}) = \frac{1}{16}$

$f(\frac{1}{16}) = \frac{1}{256}$

$f(\frac{1}{256}) = \frac{1}{65536}$

$\ldots$

Iterates are approaching 0.

Another example

Let $f(x) = -x$ and start with $x = 1$.

$f(1) = -1$
$f(-1) = 1$
$f(1) = -1$
$f(-1) = 1$
$\ldots$

Iterates are not settling down on a value.

Coloring by convergence

Color each point according to how fast it converges.

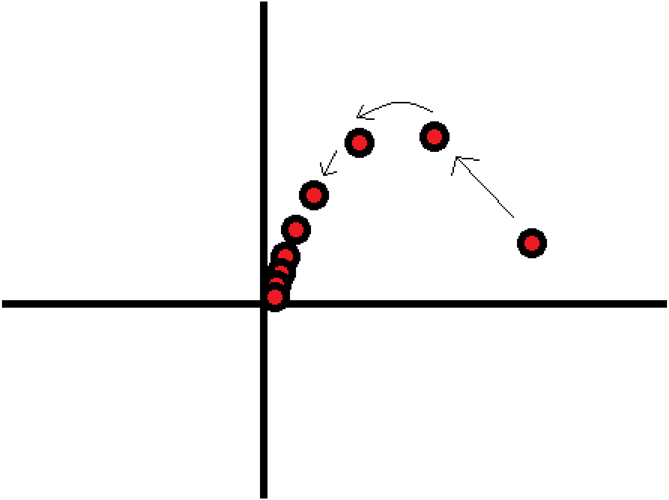
Count how many iterations until two successive values are within .00001 of each other.

Assign each count a color.

Convergence to infinity is still convergence (color by # of steps to exceed $\pm 10^5$).

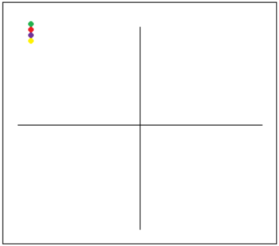
Iteration with complex numbers

Plug $z = x + iy$ into $f(z)$. Get a value, and plug that value into the function. Then plug the result of that into the function, etc.



The process

Look at all the possible starting values in a region.

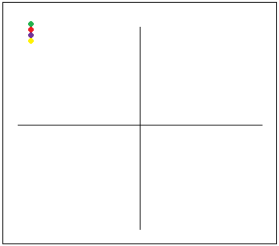


For each starting point, iterate the function.

If two successive values are within .00001 of each other, there's a very good chance that the iterates will converge.

The process, continued

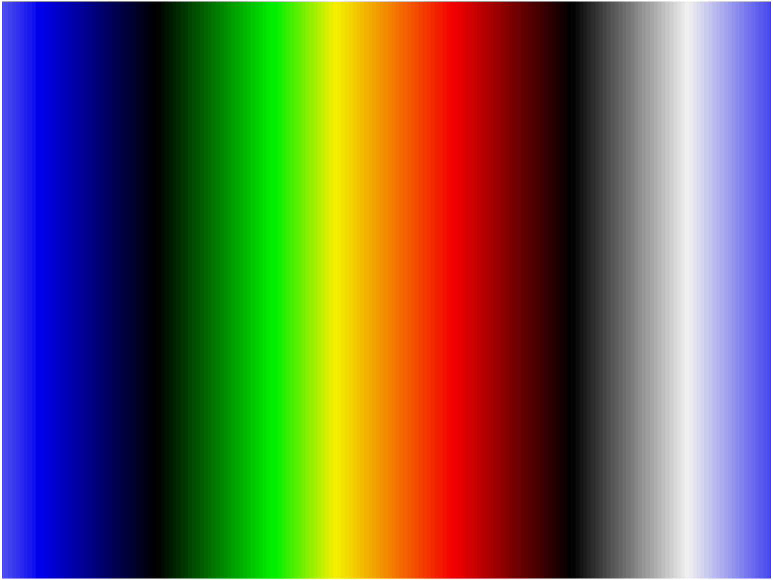
In this case, color the point with a color representing how long it took for this to happen.



It is possible that the iteration may never stop. Give up after a few hundred iterations and color the point yellow.

Note: convergence to infinity is still convergence (color by how many steps for iteration to exceed $\pm 10^5$).

Color scheme



A fractal from Newton's method

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- So, we iterate $x - \frac{x^5-1}{5x^4}$.
- We do know the roots already: $x = 1$ is the only real root.
- All of them: $\cos\left(\frac{2\pi i}{5}\right) + i \sin\left(\frac{2\pi i}{5}\right)$ for $i = 0, 1, 2, 3, 4$.

Number of iterations

Demo time!

Newton's method on $x^5 - 1$

- Iterating $x - \frac{x^5-1}{5x^4}$ to find roots.

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- $x = 10$: 8.00, 6.40, 5.12, 4.10, 3.28, 2.62, 2.10, 1.69, 1.38, 1.16, 1.04, 1.00264, 1.0000140, 1.00000000039

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- $x = .1$: 2000, 1600, 1280, 1024, [33 more iterations. . .], 1.000956, 1.0000018

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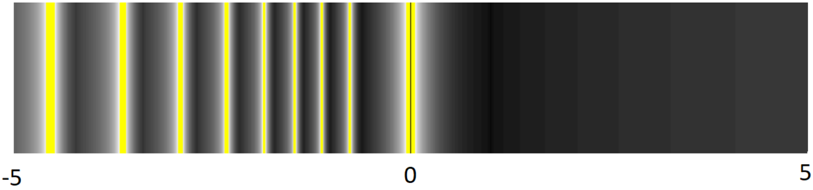
- $x = .1$: 2000, 1600, 1280, 1024, [33 more iterations. . .], 1.000956, 1.0000018
- Negatives are funny. The number of iterations to get within 10^{-5} of root at 1:

  $x = -1$: 5 iterations
  $x = -1.11$: 89 iterations
  $x = -1.5$: 28 iterations
  $x = -2$: 16 iterations
  $x = -3$: 28 iterations

Let's graph this

A plot of how many iterations before convergence.
Darker = less, yellow means ≥ 50



What about the complex roots?

- Try a complex starting value: $x = 0.2 + 0.8i$:
  Takes 4 iterations

$$0.401 + 0.999i$$
$$0.327 + 0.948i$$
$$0.309 + 0.950i$$
$$0.3090169943749474 5 + 0.9510565162951535i$$

This finds $\cos\left(\frac{2\pi}{5}\right) + \sin\left(\frac{2\pi}{5}\right) i$.

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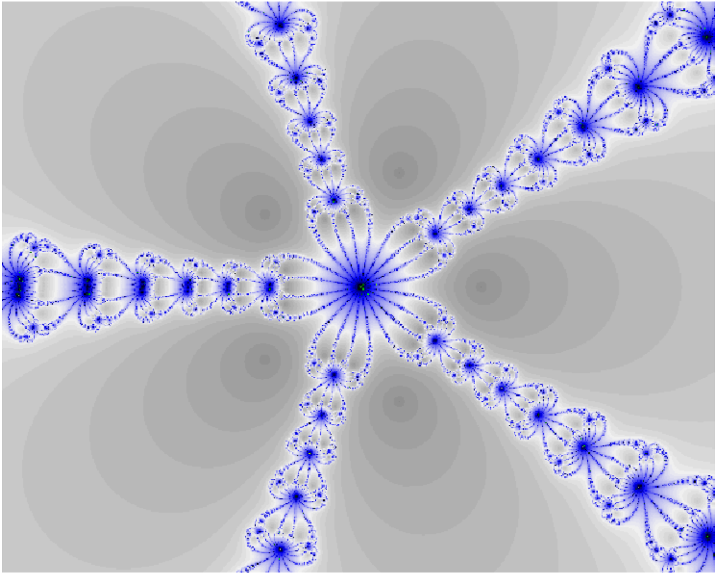
  $$\begin{aligned} &0.401 + 0.999i \\ &0.327 + 0.948i \\ &0.309 + 0.950i \\ &0.30901699437494745 + 0.9510565162951535i \end{aligned}$$

  This finds $\cos\left(\frac{2\pi}{5}\right) + \sin\left(\frac{2\pi}{5}\right) i$.

- On the other hand, $.573 + .46i$ takes 41 iterations.

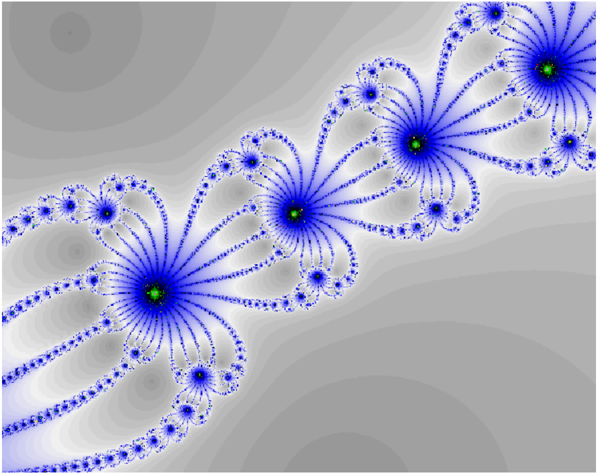
Let's graph it.



Fractal structure

Demo time!

Fractal structure



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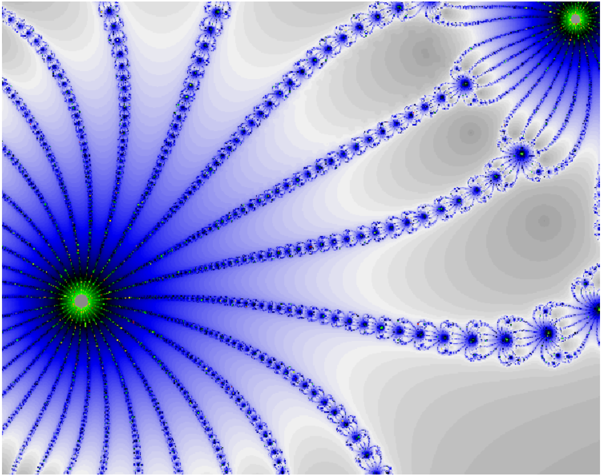

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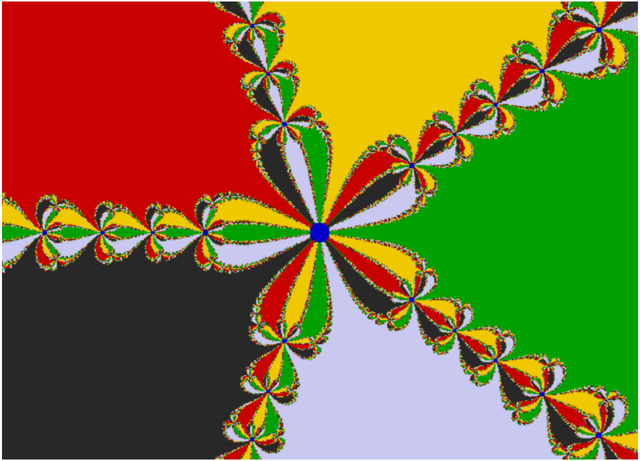
- Newton's method is a type of fixed point iteration. The roots are the fixed points and are attracting.
- Iterates pulled towards them.
- Sometimes a fight breaks out between two roots, and sometimes they both lose.
- The next picture shows what root each starting point is attracted to.

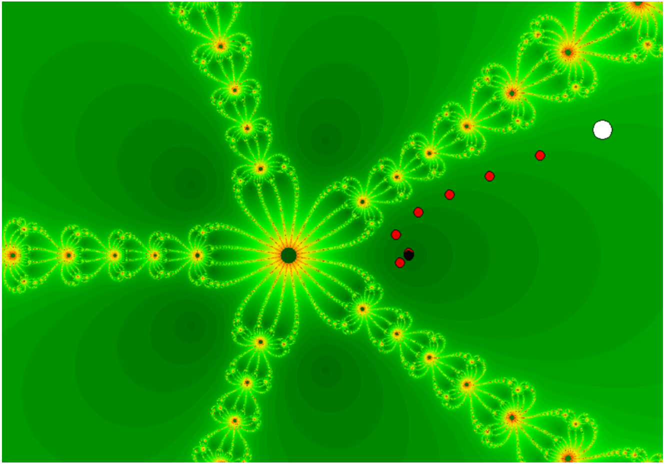
Coloring by root



Orbits

Demo time!

A particular orbit



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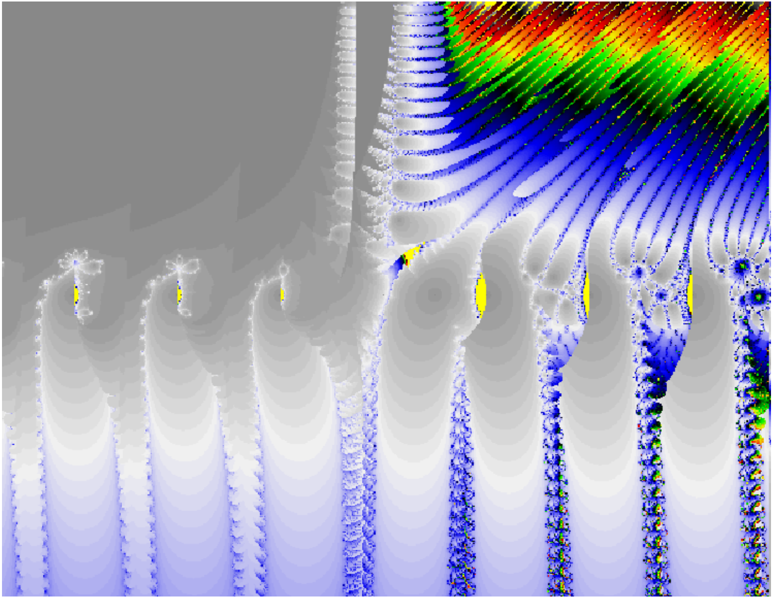
- For each value of c, imagine doing the plot but using a single point on the plot to represent the entire picture.
- We'll use $z = -1$ as the representative point.

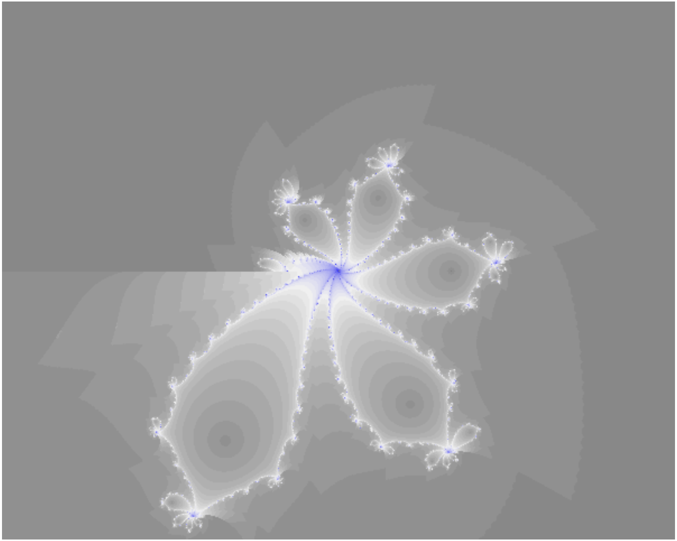
Newton index set

Demo time!

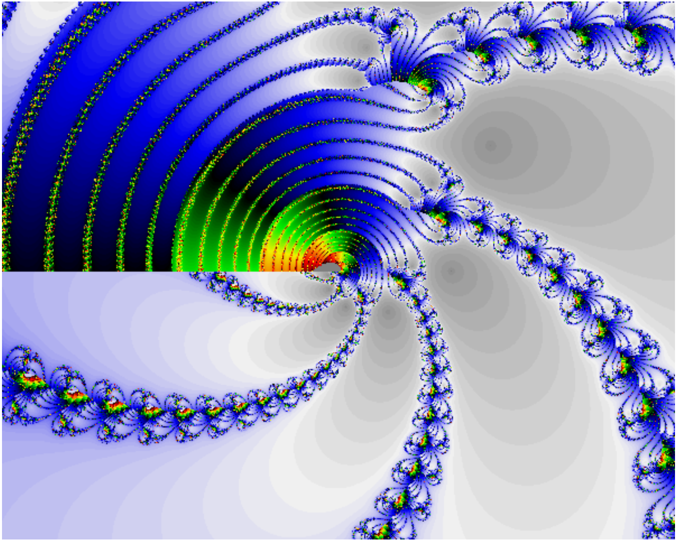
Newton Index Set



Newton, $c \approx -5.4 + 1.5i$



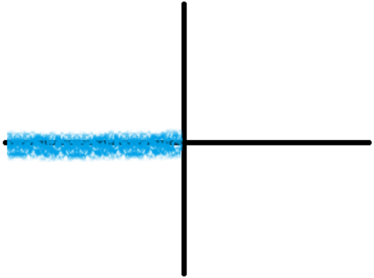
Newton, $c \approx 5.475 + 4.45i$

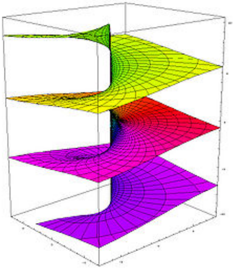


The complex logarithm

$\ln z = \ln |z| + i \arg z$

Take branch where $-\pi < \arg z \leq \pi$.





From http://www.answers.com/topic/branch-point

Complex exponentiation

- $\ln z = \ln |z| + i \arg z$
- $e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y)$
- $z^c = e^{c \ln(z)}$

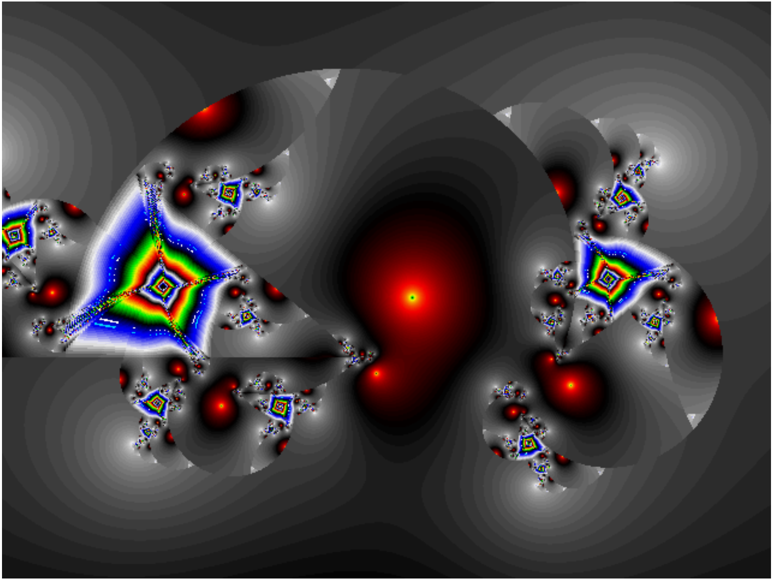
$f(z) = c \sin(\ln z)$

$\sin(x + iy) = \sin x \cosh y + i \cos x \sinh y$

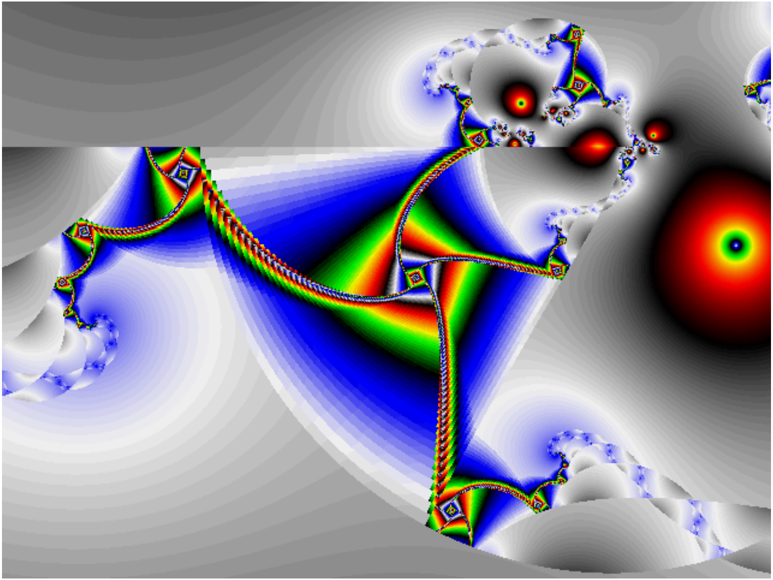
$\ln z = \ln |z| + i \arg z$

Different values of c produce wildly different pictures.

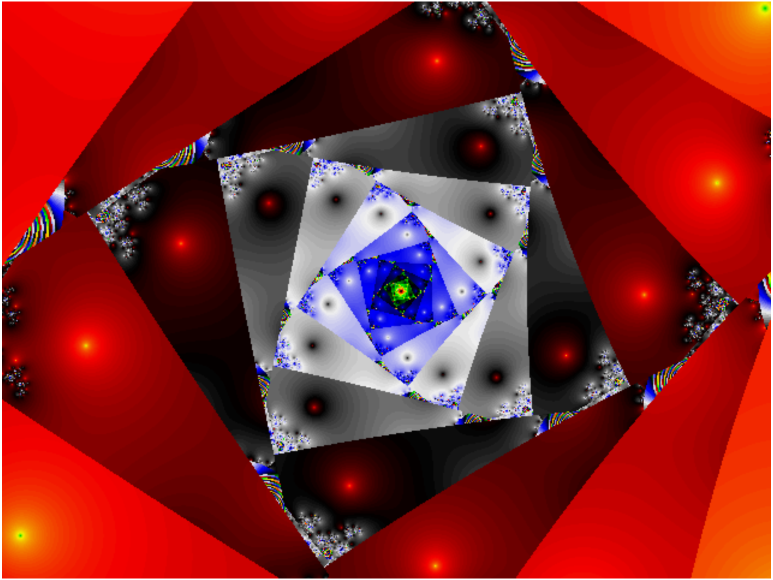
$f(z) = c\sin(\ln z)$, $c = .01 + .99i$



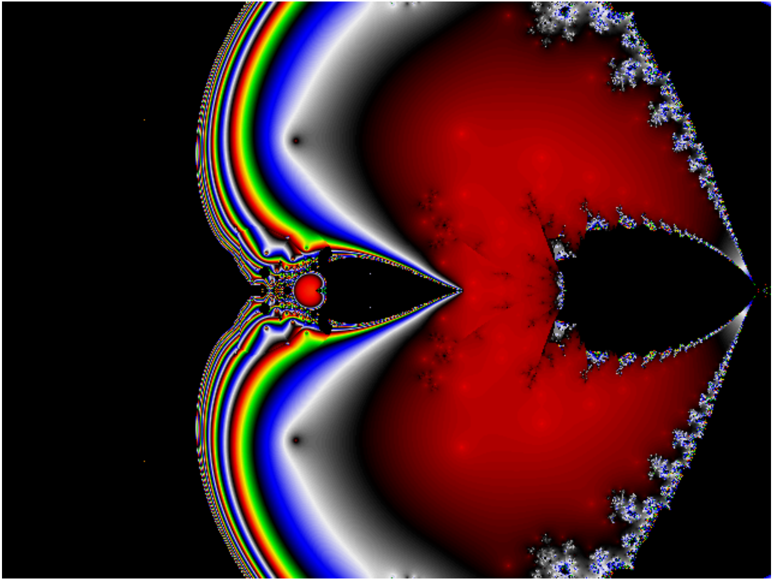
$c\sin(\ln z)$, $c = -1 + 2.25i$



$c \sin(\ln z)$, $c = 2.29 - 6.55i$

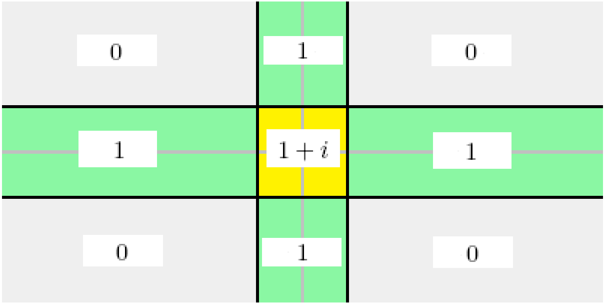


Index set for $c \sin(\ln z)$

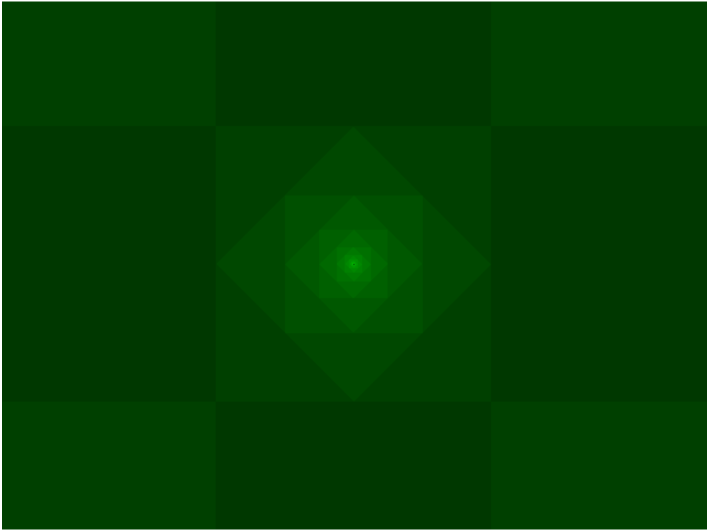


An interesting piecewise function

Call it $\gamma(z)$.



$f(z) = z\gamma(z)$



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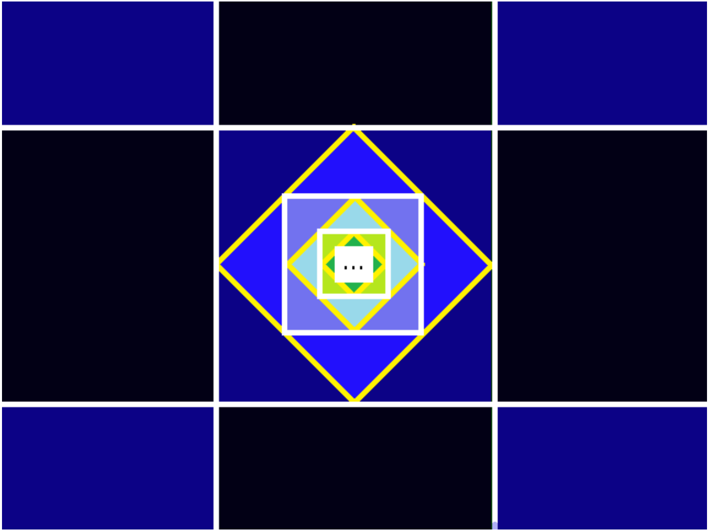
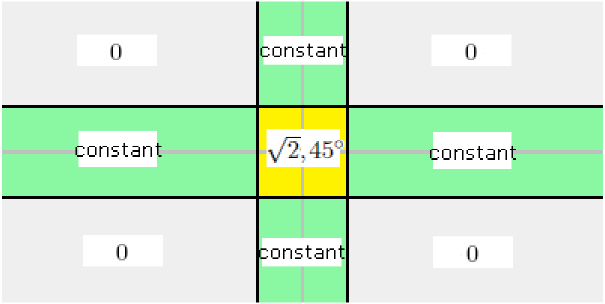


$f(z) = z\gamma(z)$

Given $z = re^{i\theta}$, $f(z)$ is described by

$$\begin{cases} r \mapsto \sqrt{2}r, \quad \theta \mapsto \theta + 45^\circ & \text{center box} \\ r, \theta \text{ constant} & \text{strips} \\ r, \theta \mapsto 0 & \text{elsewhere} \end{cases}$$



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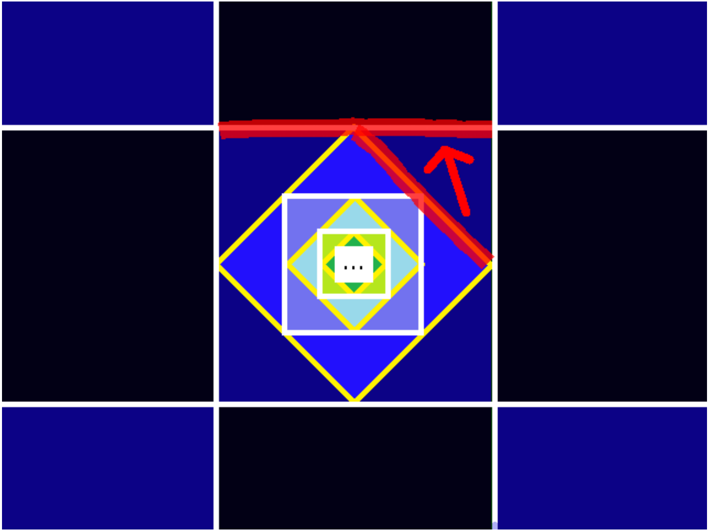

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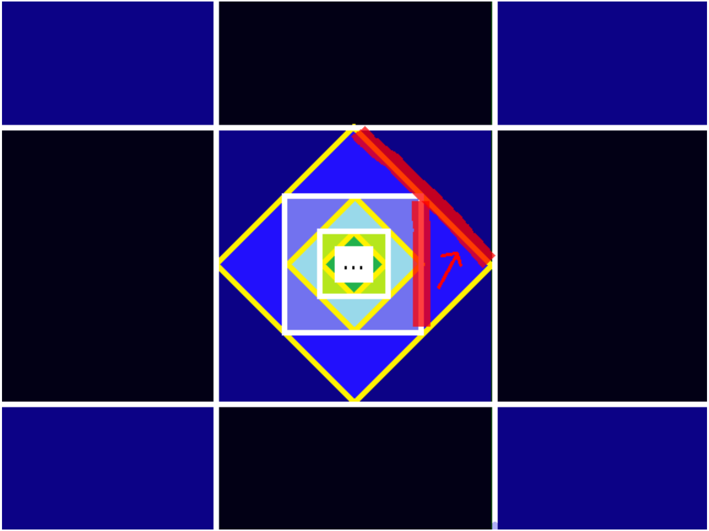

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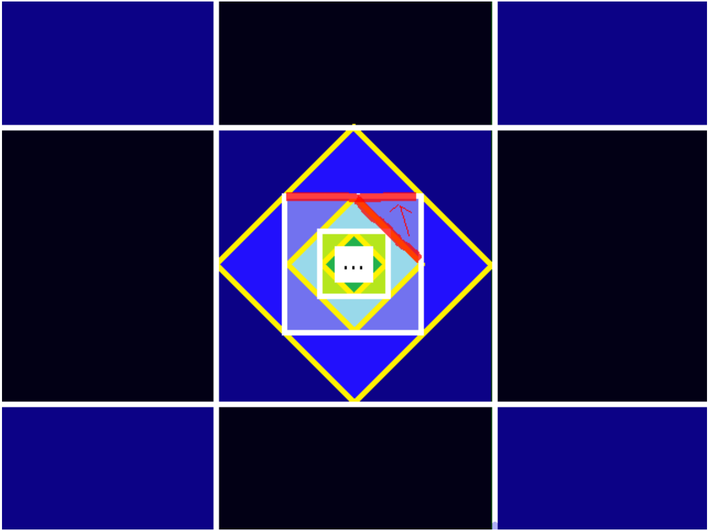
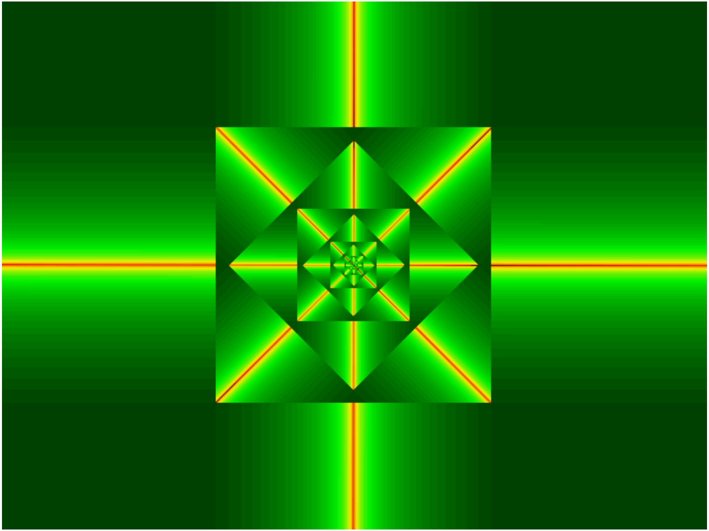


Orbits

Demo time!

$c = 1.1$



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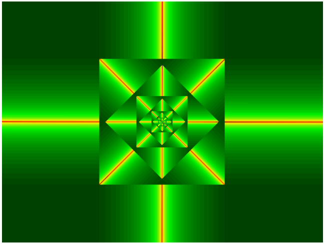
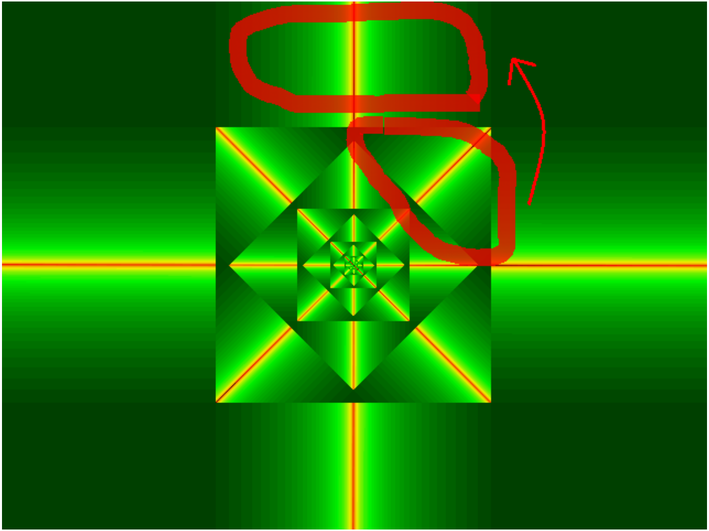
$f(z)$ is described by:

$$\begin{cases} (1.1\sqrt{2}, \quad 45^\circ) & \text{center box} \\ (1.1, \quad 0^\circ) & \text{strips} \\ (0, \quad 0^\circ), & \text{elsewhere} \end{cases}$$



In the outside strips, the small dilation leads to slow convergence. Points within the square eventually get pushed into the outside strips.

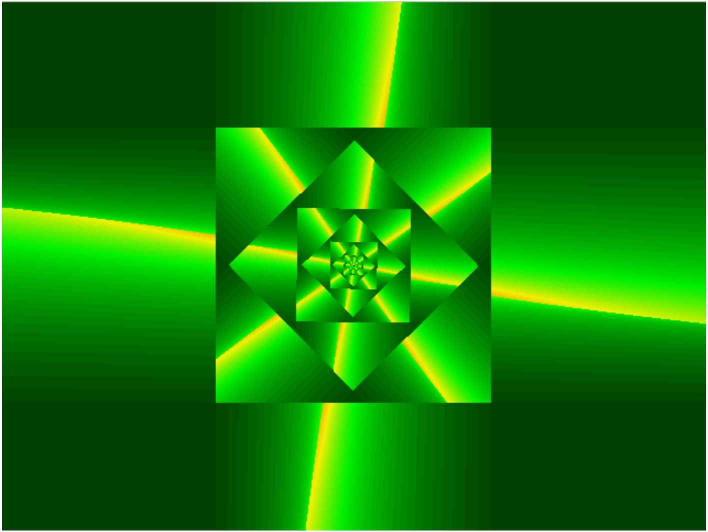


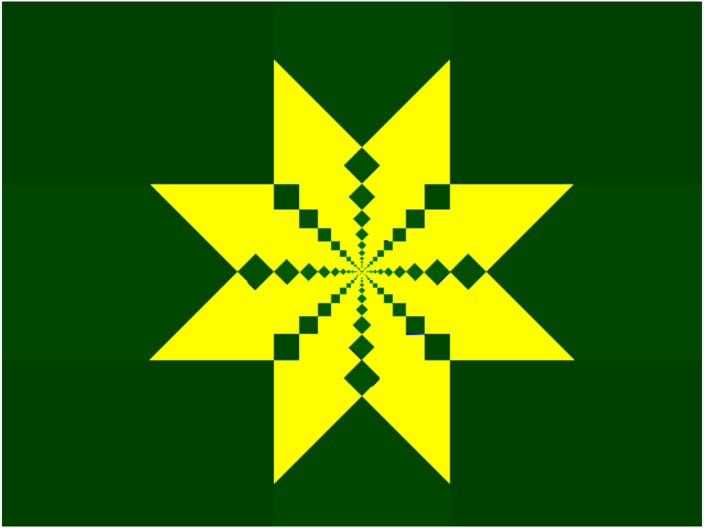
Orbits

Demo time!

$c = 1.1 + .01i$

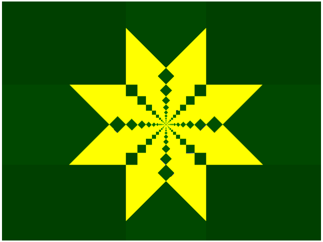
Adding a small imaginary term adds a bit of rotation, but no major change.



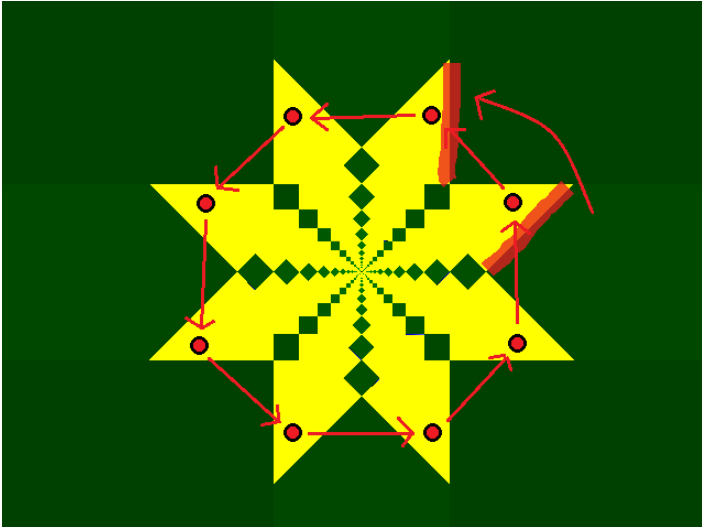


$f(z)$ is described by

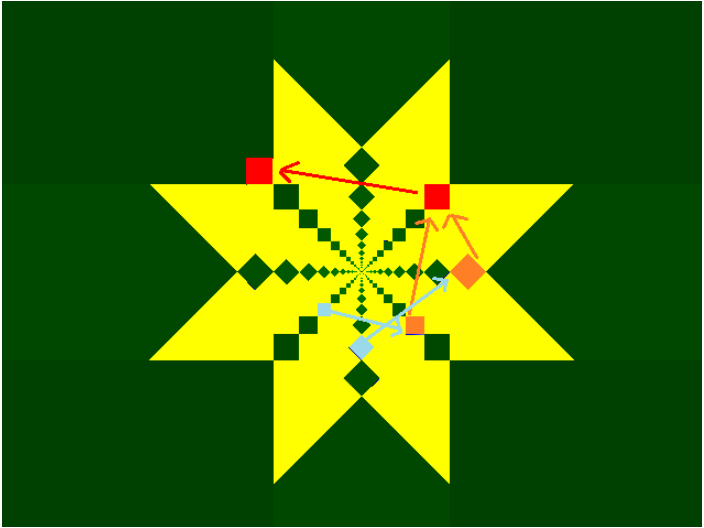
$$\begin{cases} (\sqrt{2}, \quad 90^\circ) & \text{center box} \\ (1, \quad 45^\circ) & \text{strips} \\ (0, \quad 0^\circ), & \text{elsewhere} \end{cases}$$



Many points will cycle endlessly.



Where the green boxes and diamonds come from:

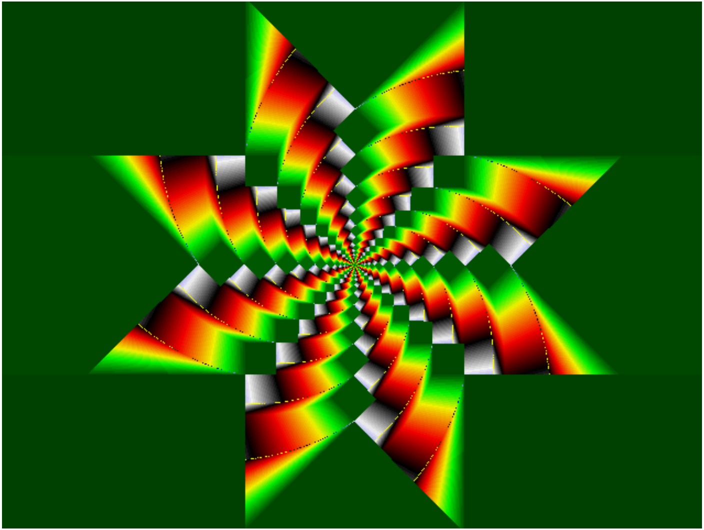


Orbits

Demo time!

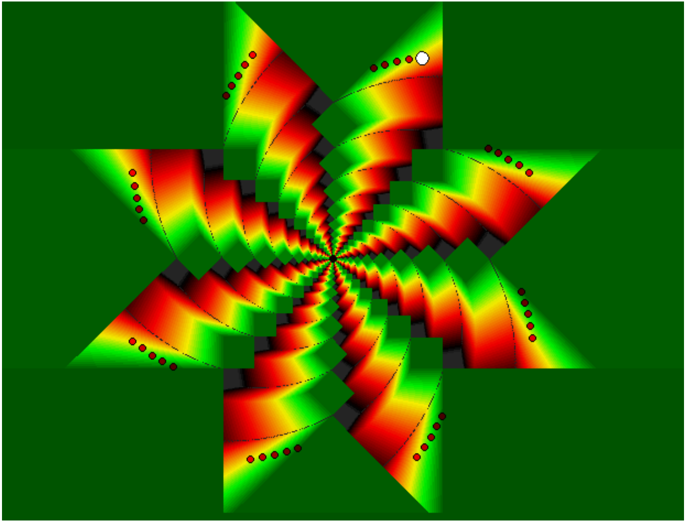
$c = .700 + .709i$

Move from $c \approx .707 + .707i$ to $.700 + .709i$.

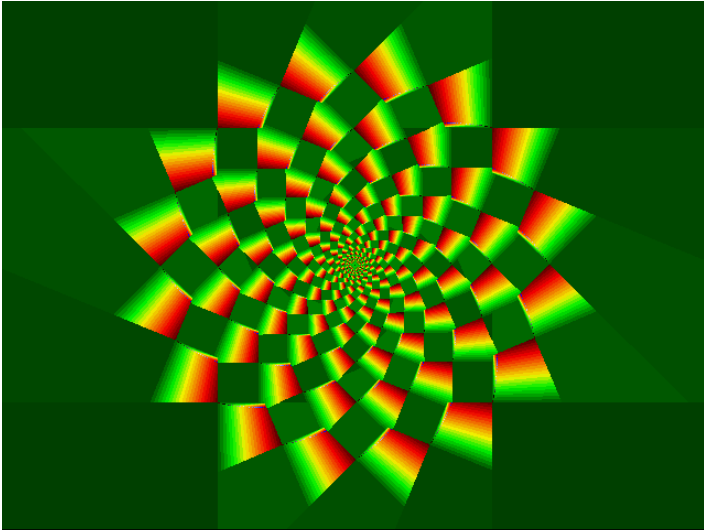


$c = .700 + .709i$

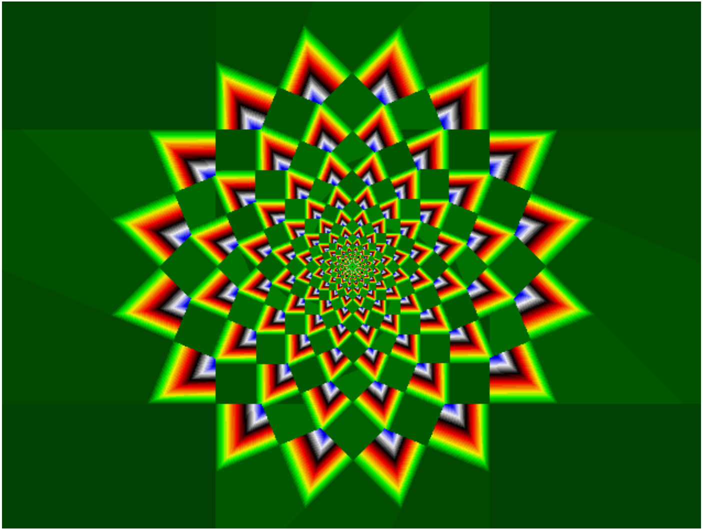
The red circles are the actual iterates. Rotation is not quite 45°. The slight perturbation adds up.

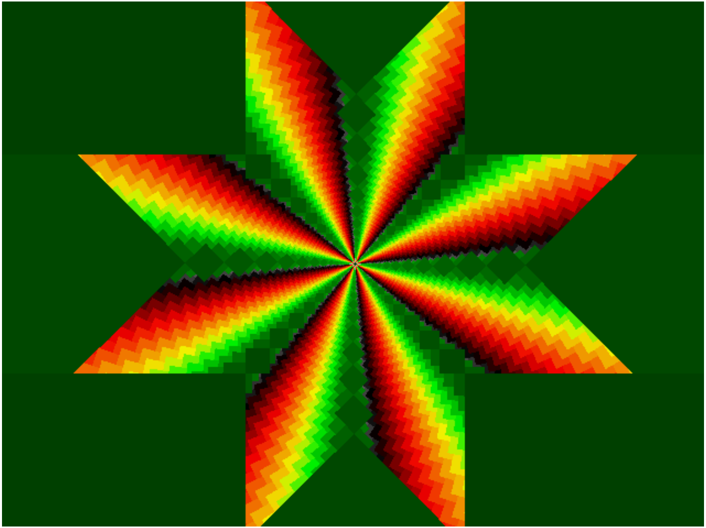


$c = .926 + .381i$

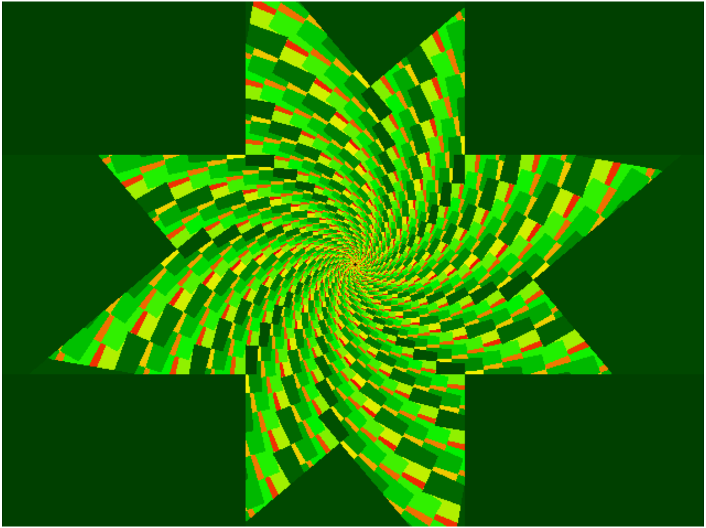


$c = .926 + .384i$

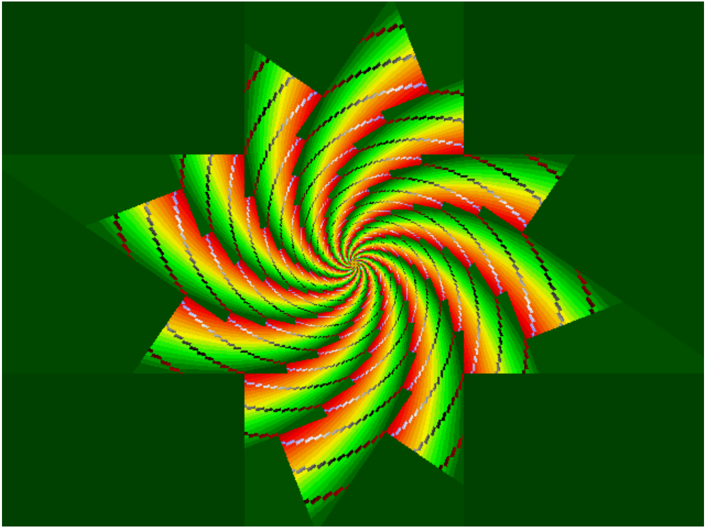




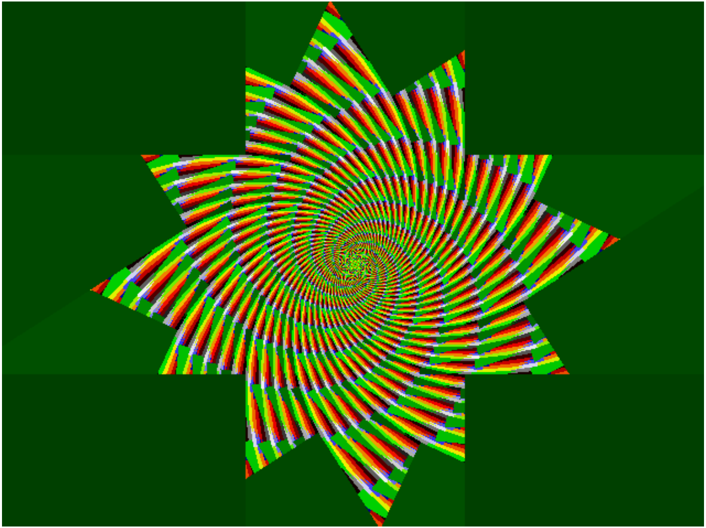
$c = .561 + .667i$



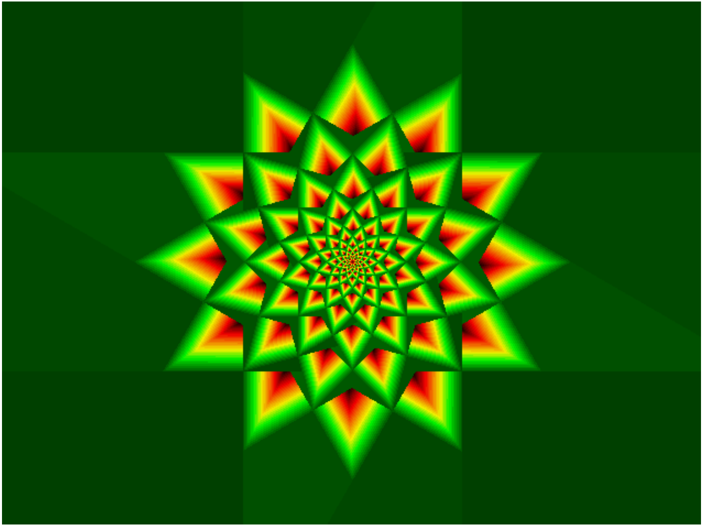
$c = .752 + .516i$



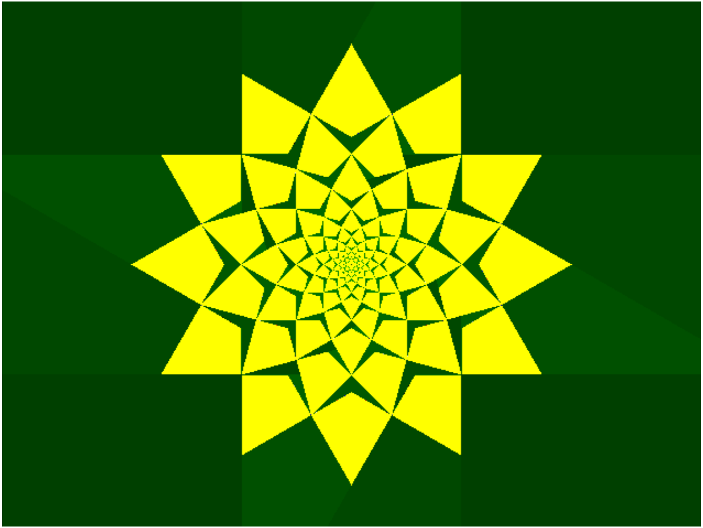
$c = .489 + .765i$



$c = .870 + .504i$

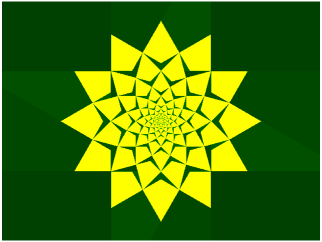


$c = \frac{\sqrt{3}}{2} + \frac{1}{2}i$



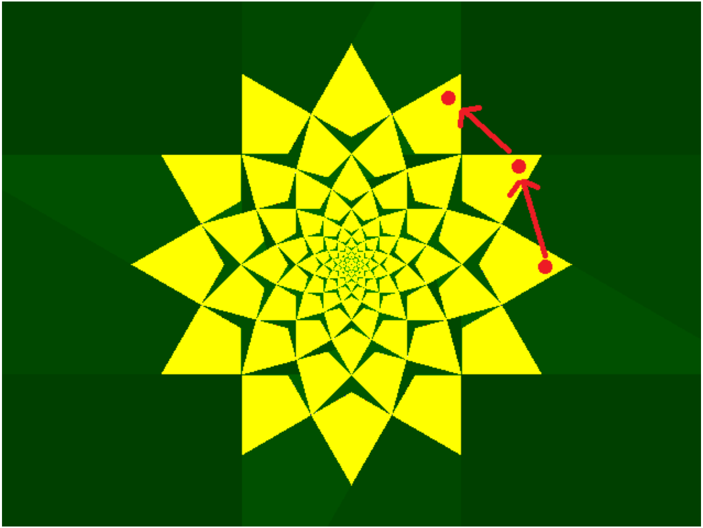
$f(z)$ is described by

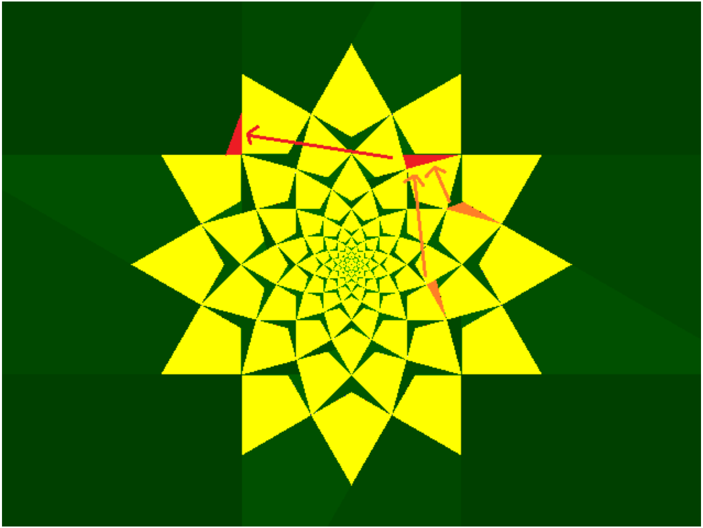
$$\begin{cases} (\sqrt{2}, \quad 75^\circ) & \text{center box} \\ (1, \quad 30^\circ) & \text{strips} \\ (0, \quad 0^\circ), & \text{elsewhere} \end{cases}$$



$c = \frac{\sqrt{3}}{2} + \frac{1}{2}i$

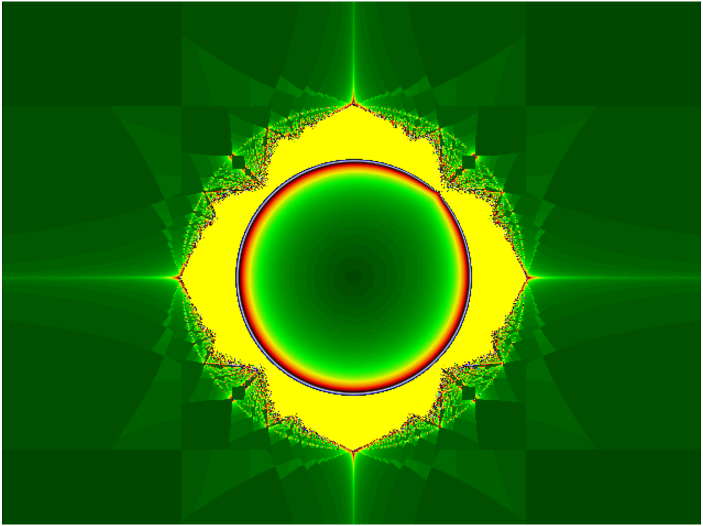
Get cycles again because 360 is divisible by 30.





Index set

For each value of c, see what color we get when we iterate starting at $z = 1$.

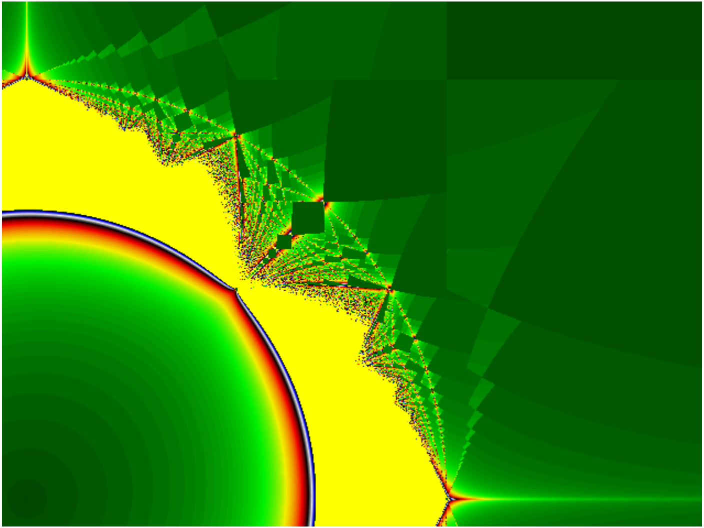


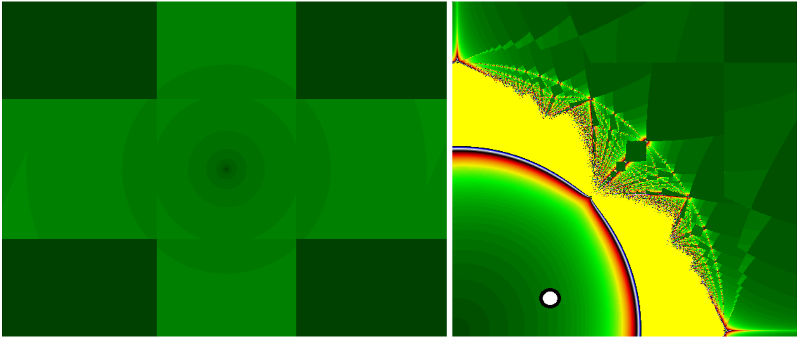
Index set

Demo time!

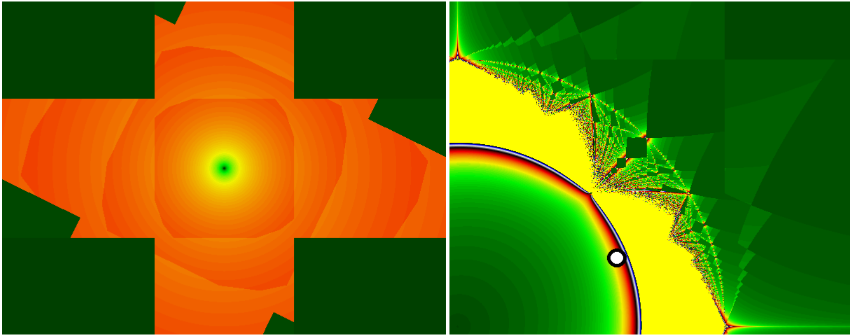
Index set close-up

Color of $z = 1$ is somewhat representative of the entire image.

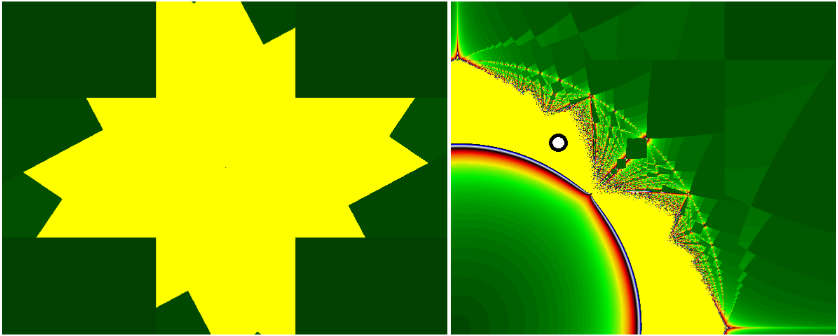




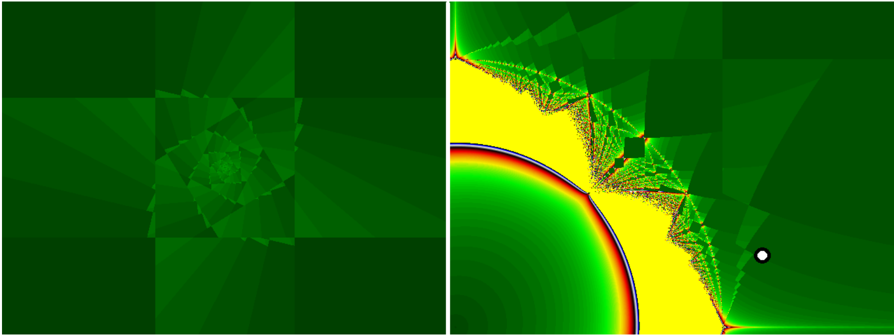
$c = .584 + .287i$



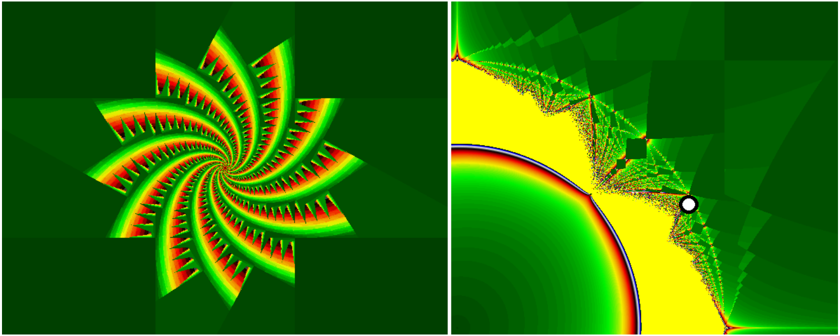
$c = .381 + .683i$



$c = .1139 + .271i$

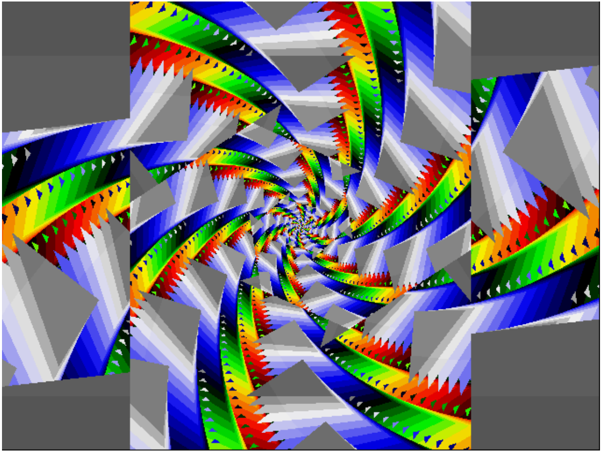


$c = .854 + .465i$



Things to try

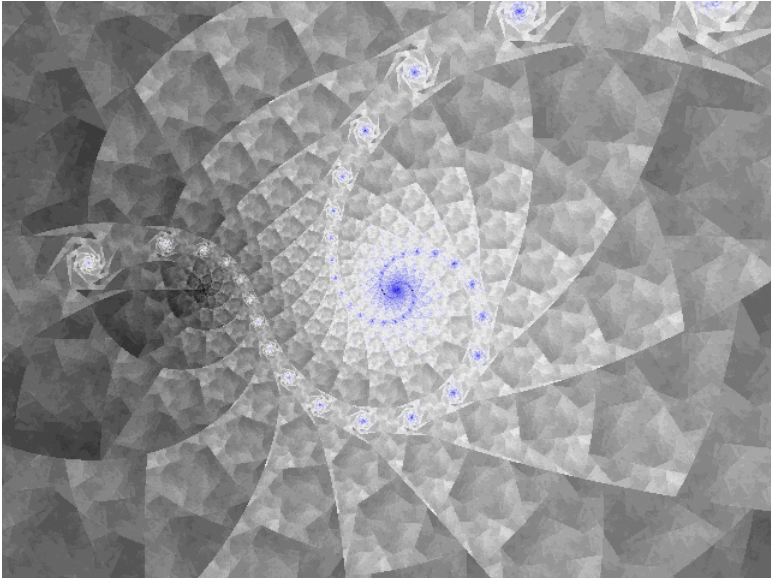
- Other piecewise functions
- Change z to z^2 or something else
- Other types of transformations



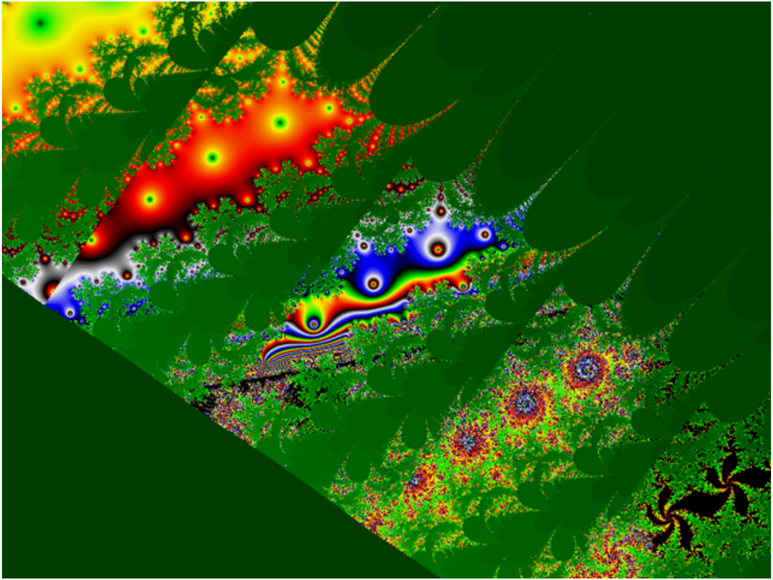
Gallery

Here follows a gallery of some interesting pictures.

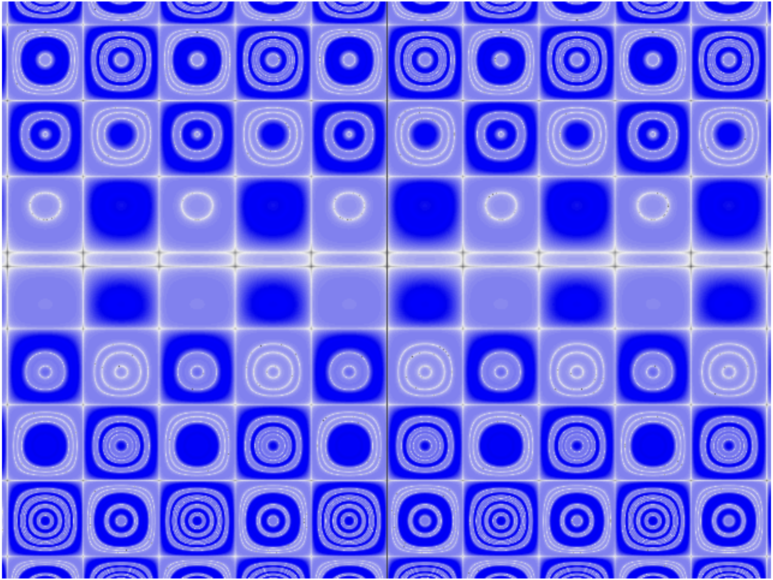
$f(z) = z^c$, $c = -1.09 + .197i$



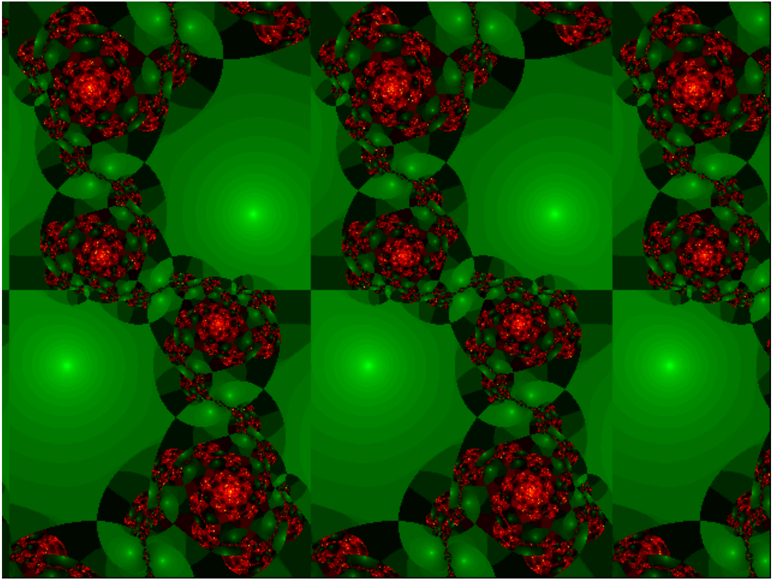
$f(z) = z^{z^c}$



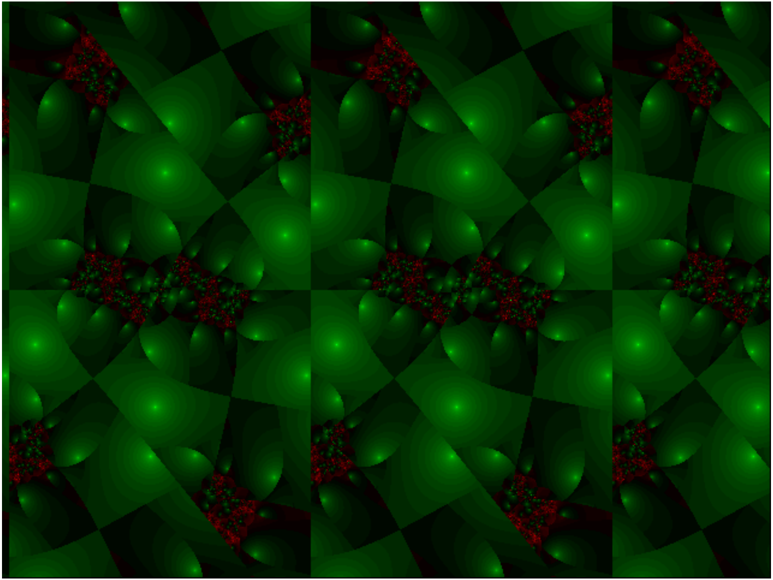
$f(x+iy) = c(\sin x)(\cos y)(1-y)$, $c = .76 - .53i$



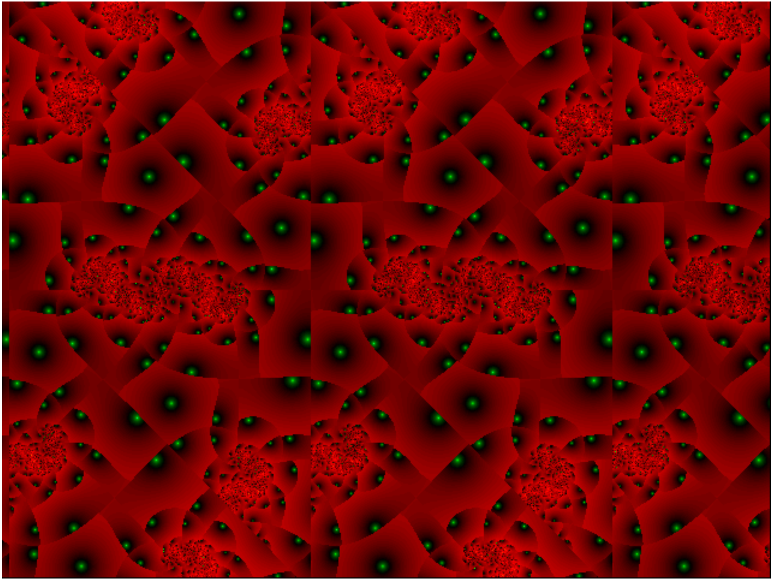
$c \cdot \ln(\sin z)$



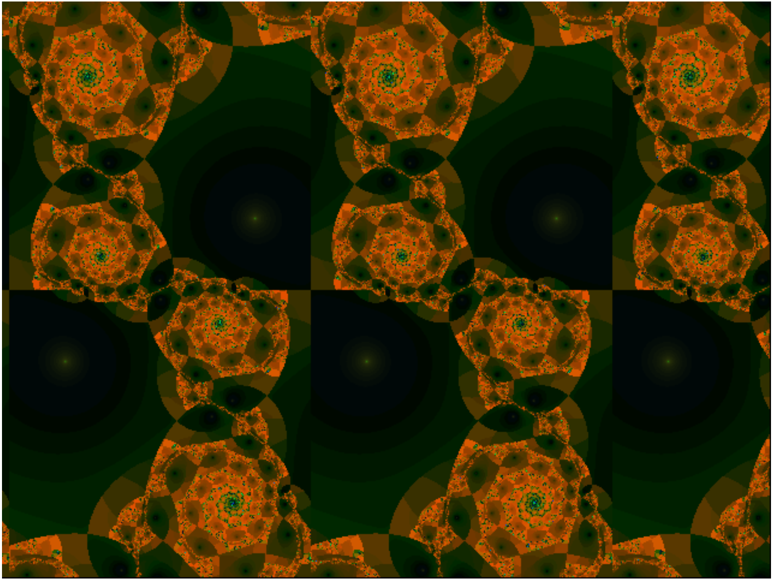
$c \cdot \ln(\sin z)$



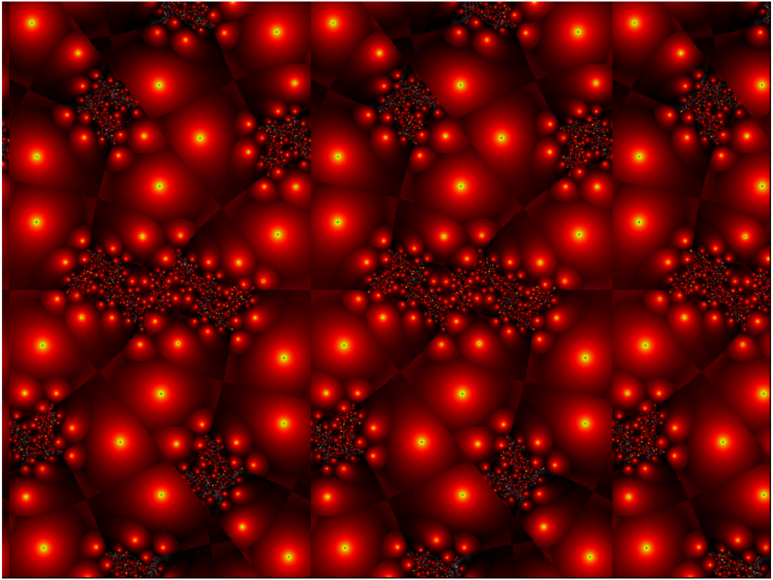
$c \cdot \ln(\sin z)$



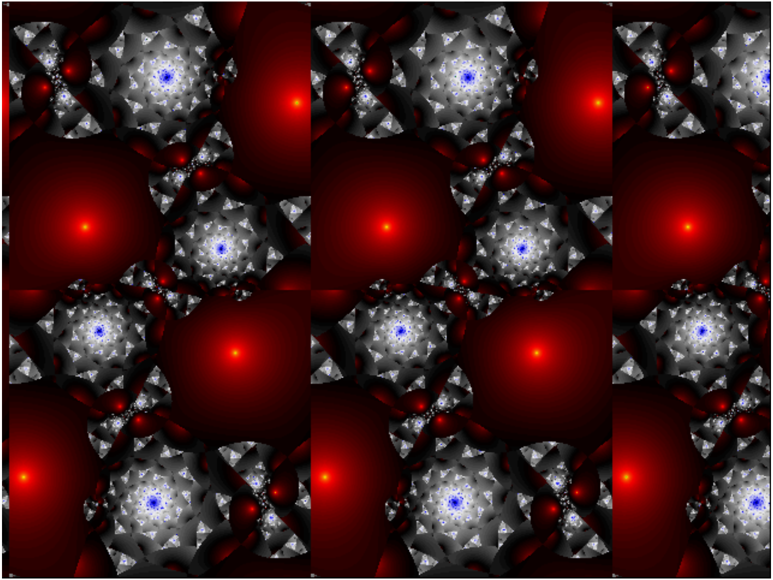
$c \cdot \ln(\sin z)$

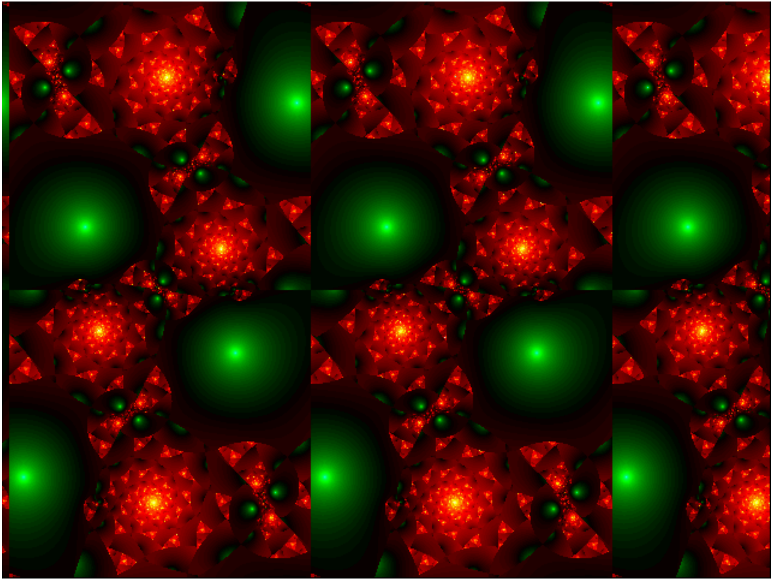


$c \cdot \ln(\sin z)$

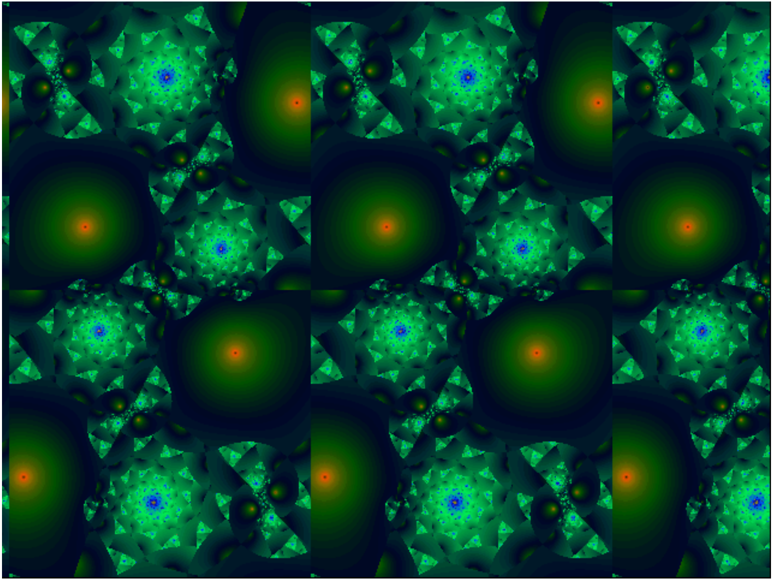


$c \cdot \ln(\sin z)$

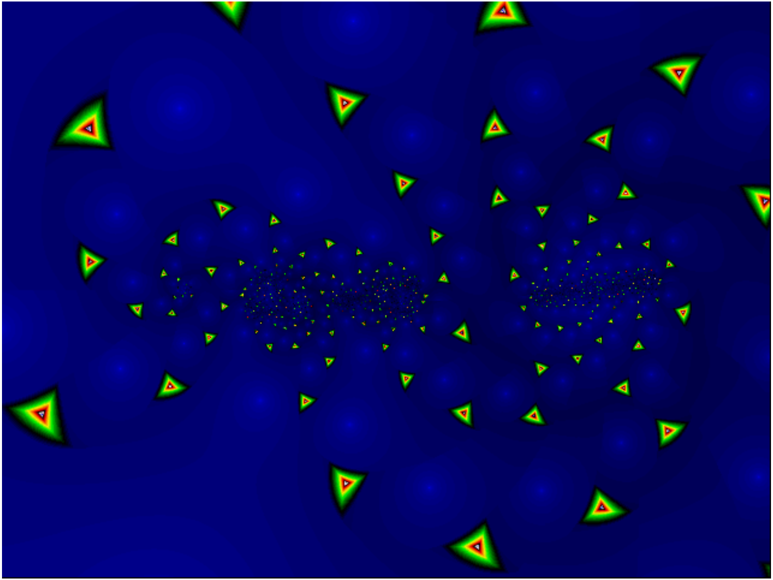




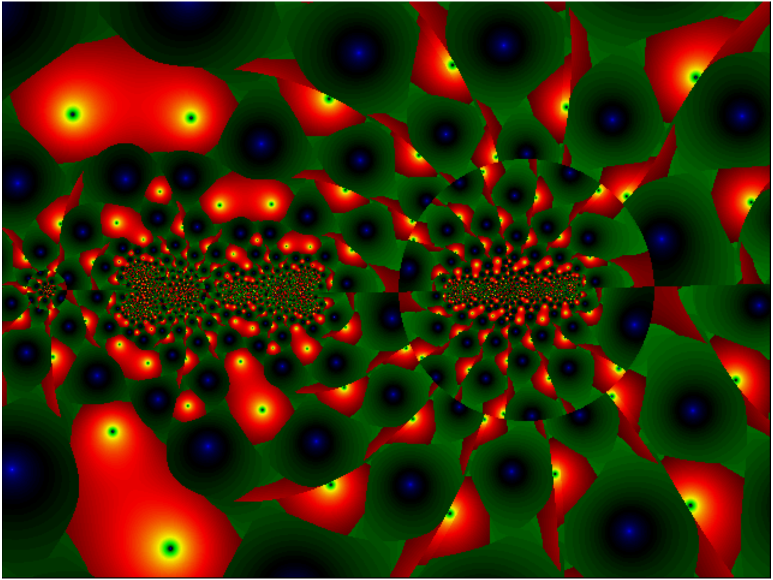
$c \cdot \ln(\sin z)$



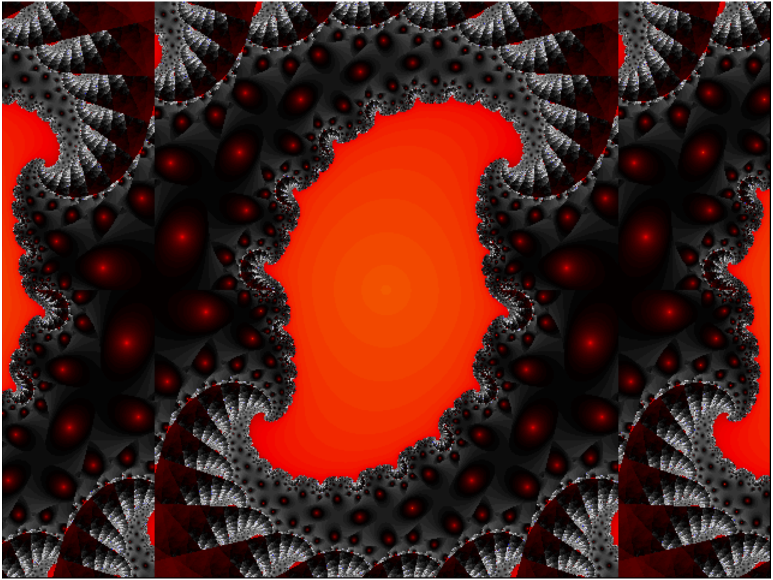
$c \cdot \sin\left(\ln\left(\sin\left(\ln z\right)\right)\right)$



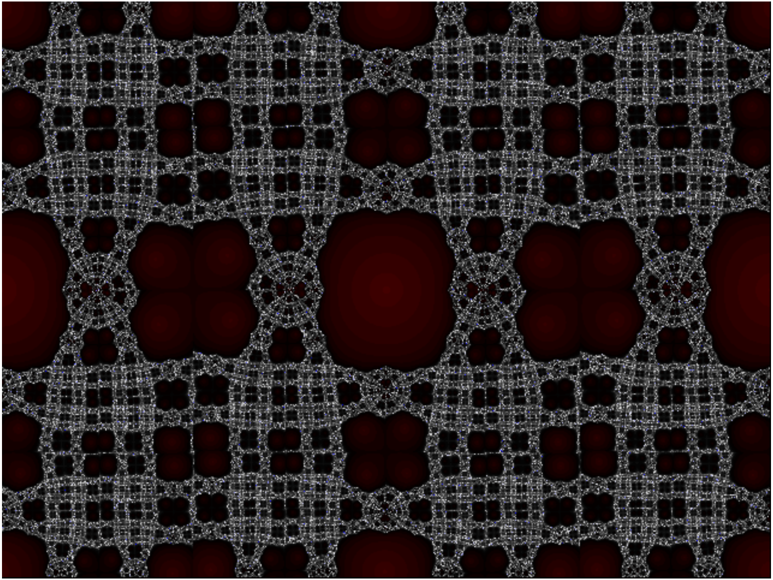
$c \cdot \sin(\ln(\sin(\ln z)))$



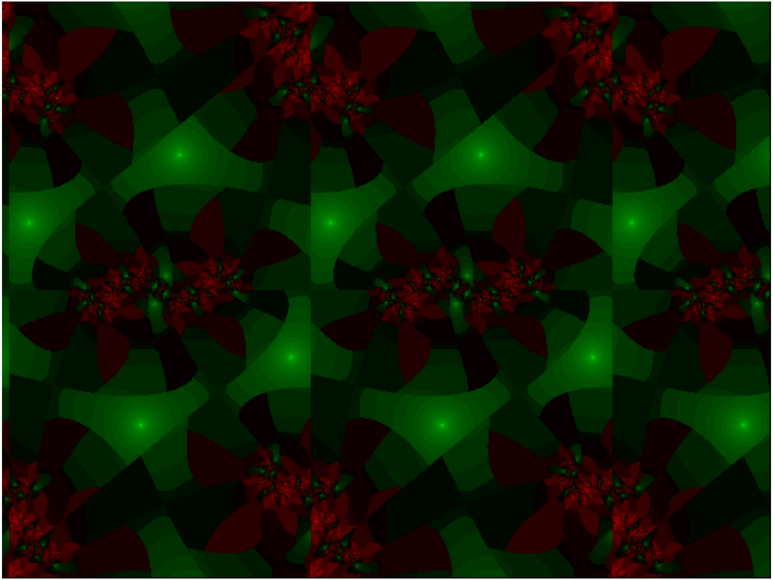
$c \cdot \ln(\cos z)$



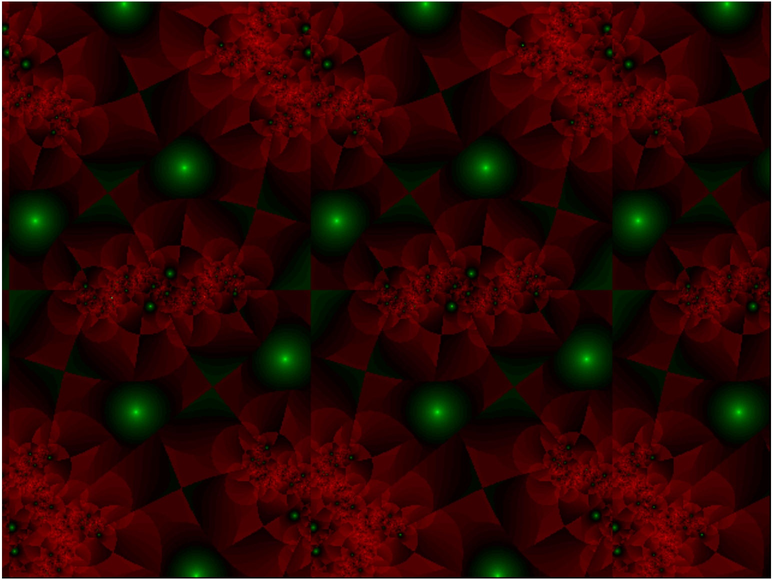
$c \cdot \ln(\cos z)$



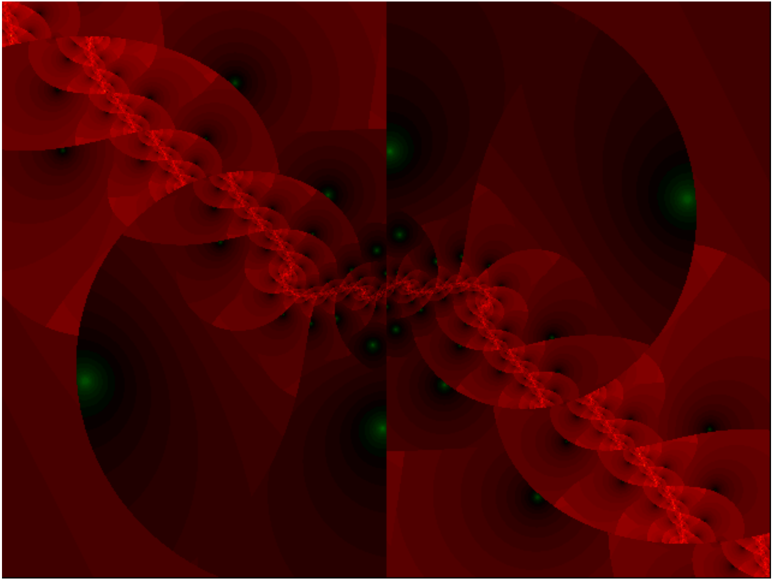
$c \cdot \ln(\csc z)$



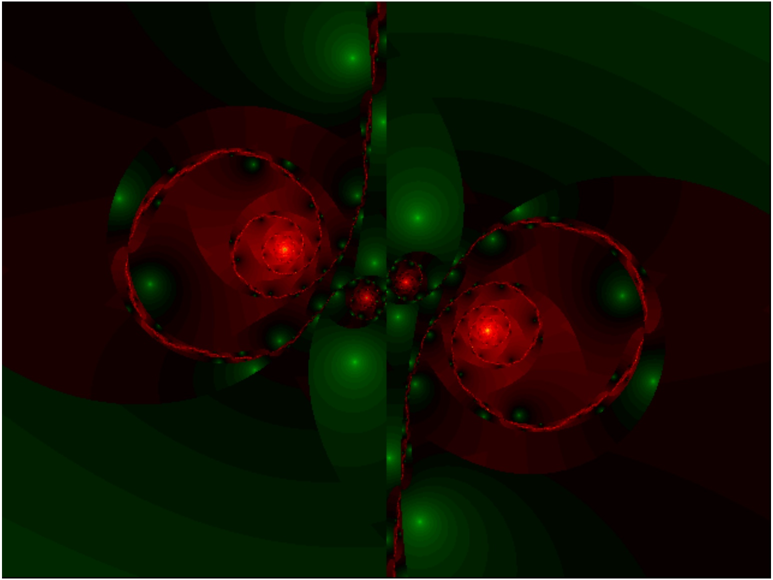
$c \cdot \ln(\csc z)$



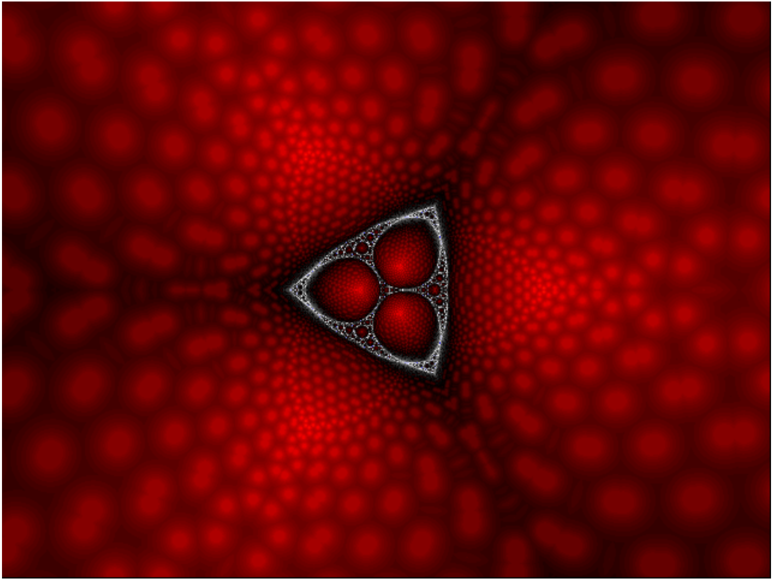
$c \cdot \ln z^4$



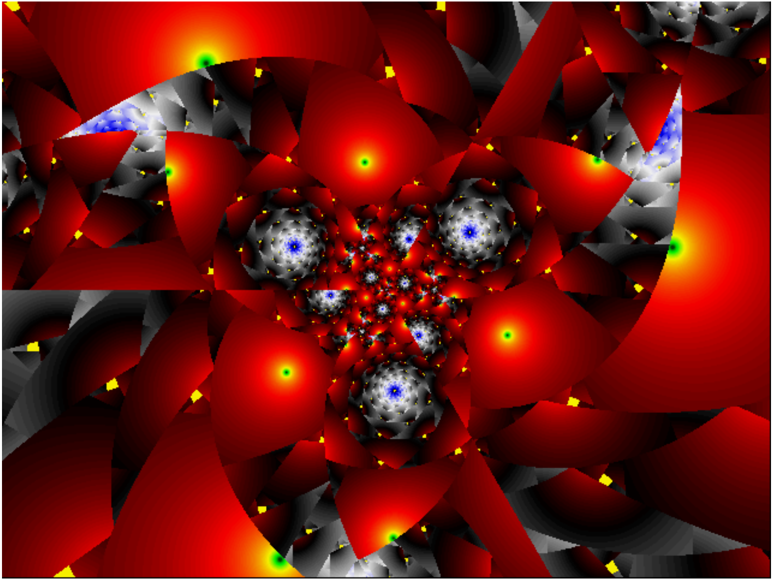
$c \cdot \ln z^2$



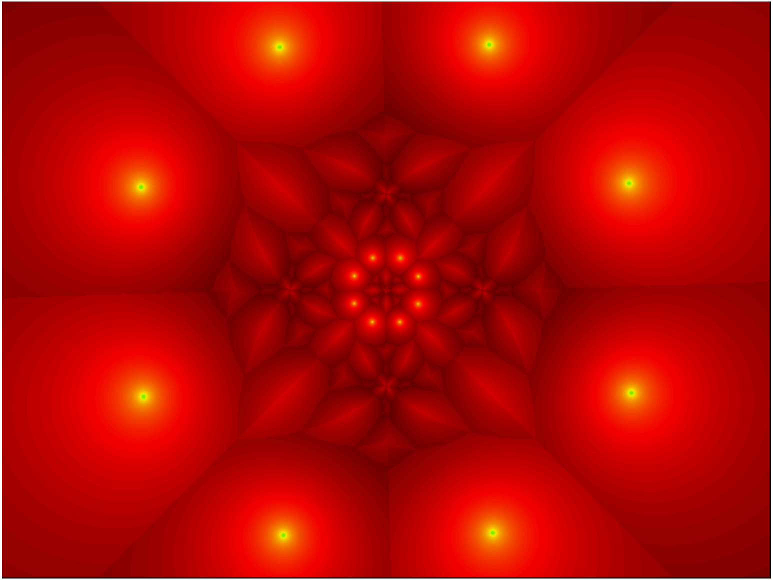
$c \cdot \ln z^3$



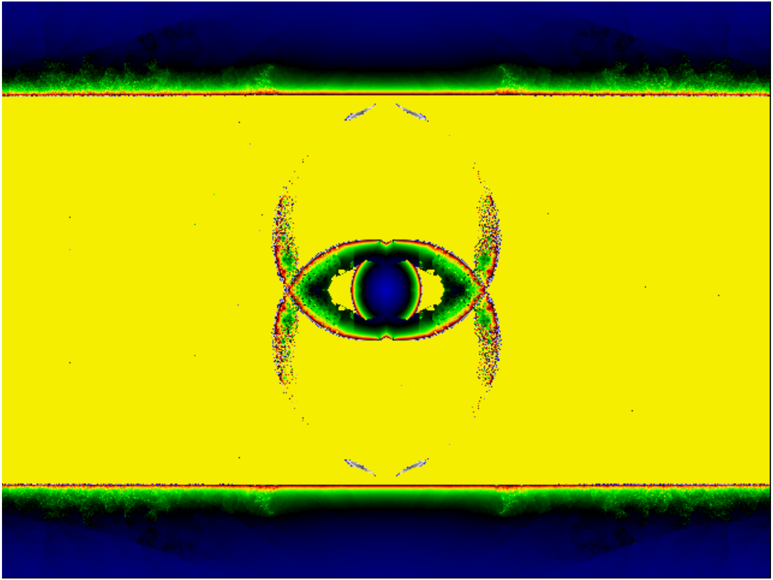
$c \cdot \ln z^3$



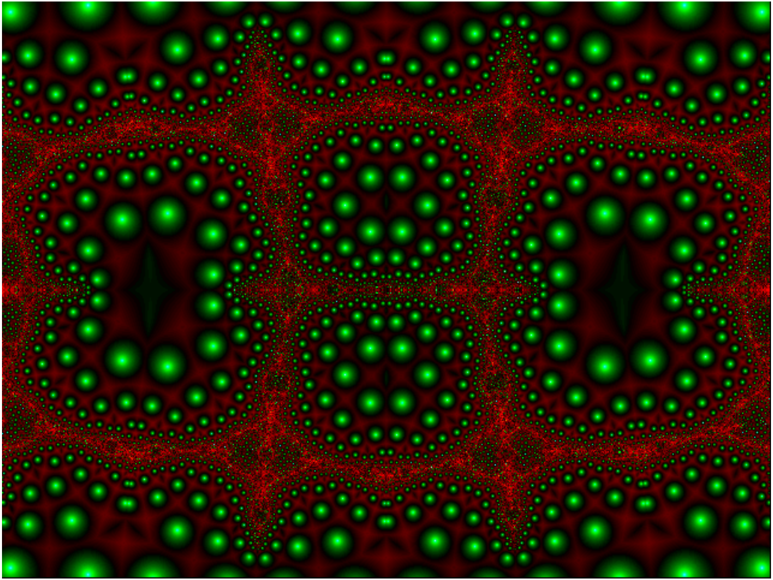
$c \cdot \ln z^4$



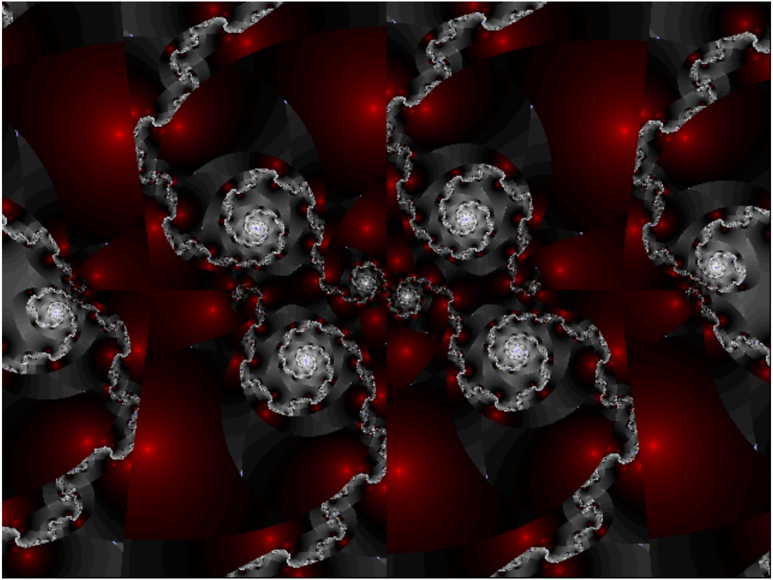
$c \cdot \ln(z \cdot \sin z)$



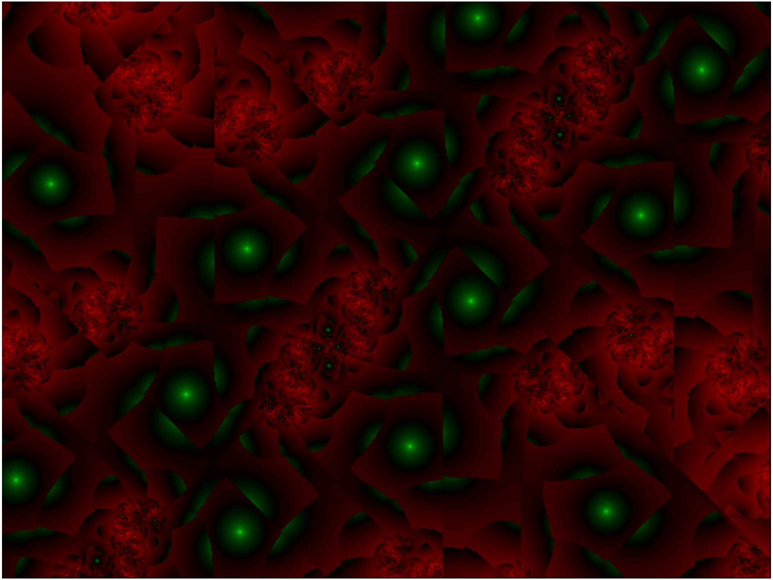
$c \cdot \ln(z \cdot \sin z)$



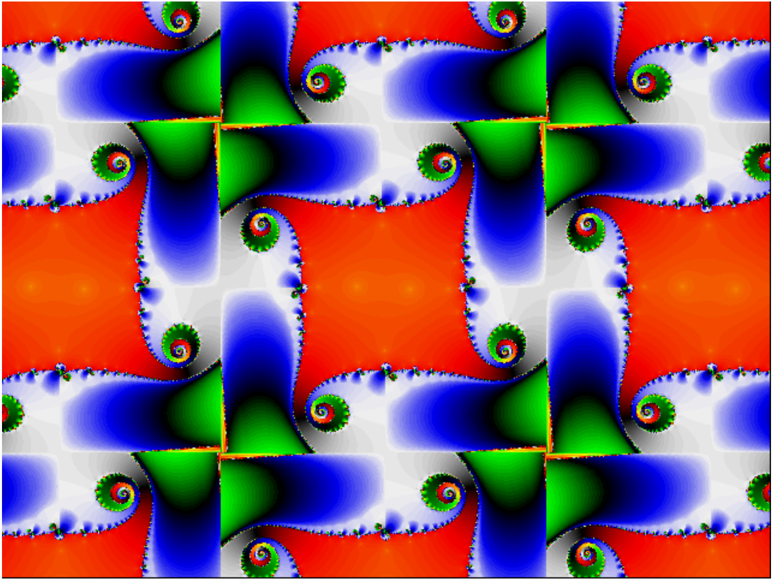
$c \cdot \ln(z \cdot \sin z)$



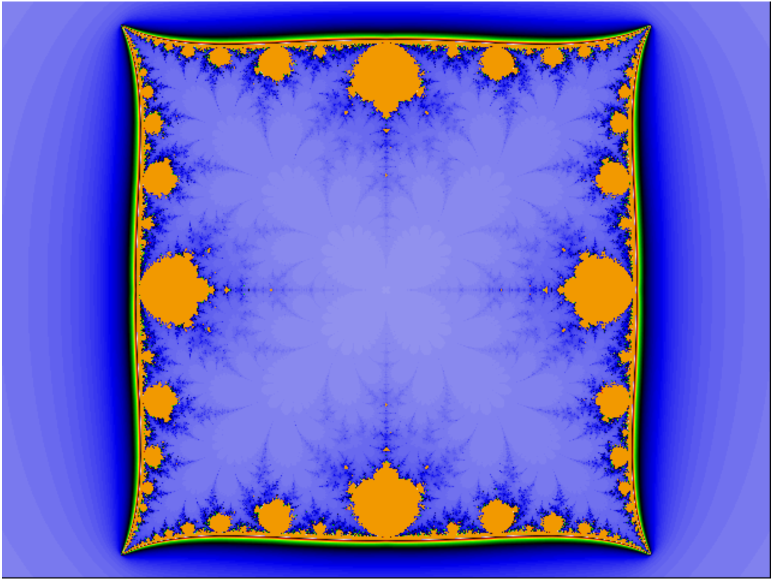
$c \cdot \ln(z \cdot \sin z)$



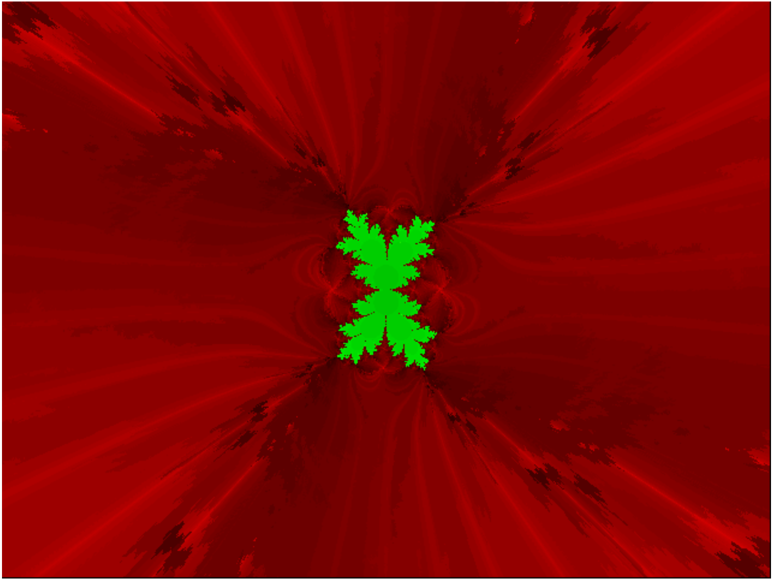
$c \cdot \ln\left(\cos\left(z + c\right)\right)$

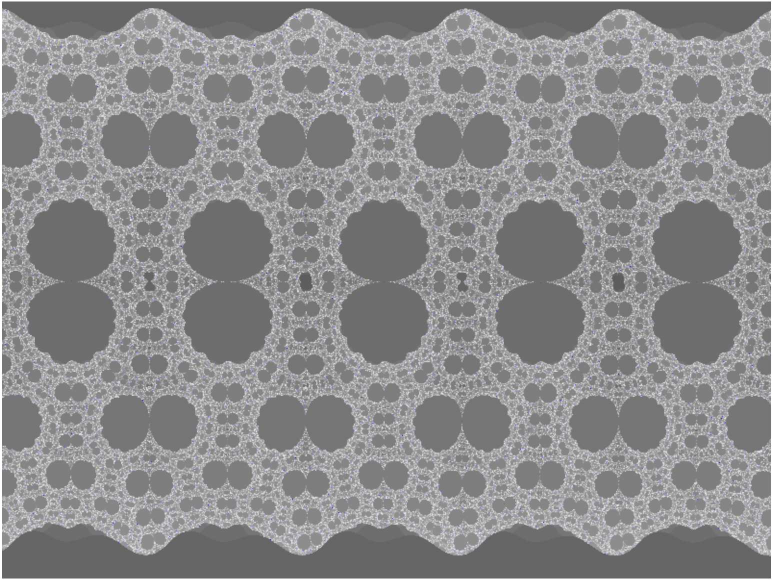


$c \cdot \sec\left(1/z^2\right)$

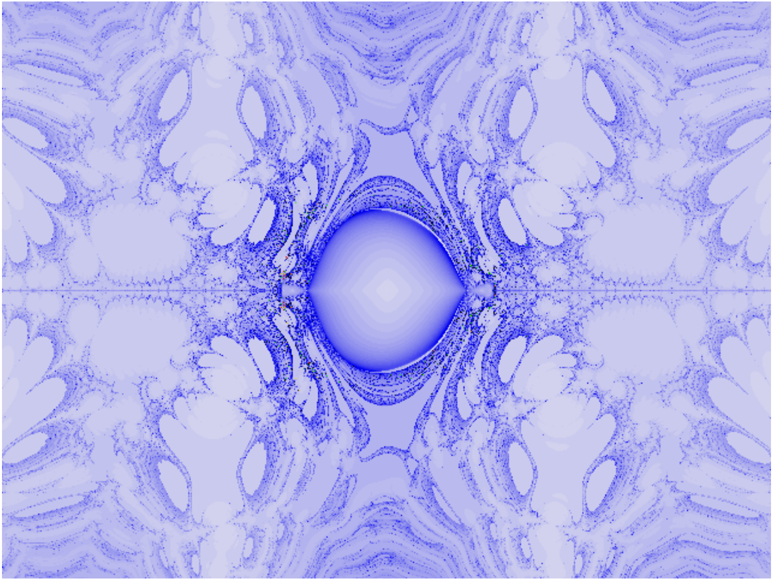


$c \cdot \csc(1/z)$

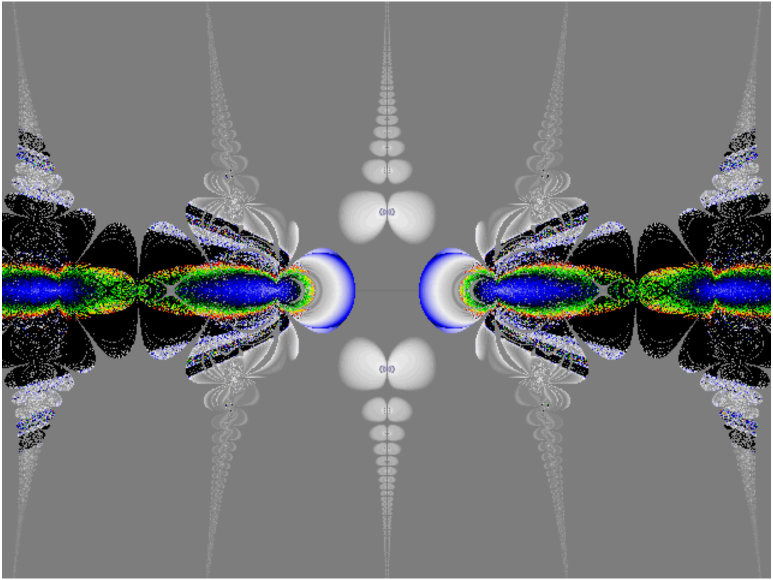




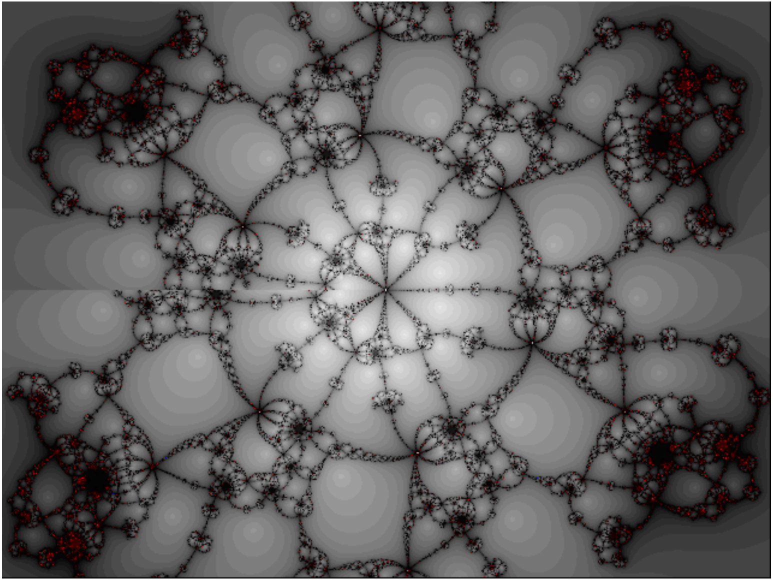
$|z/(\cos(c \cdot \sin z))|$



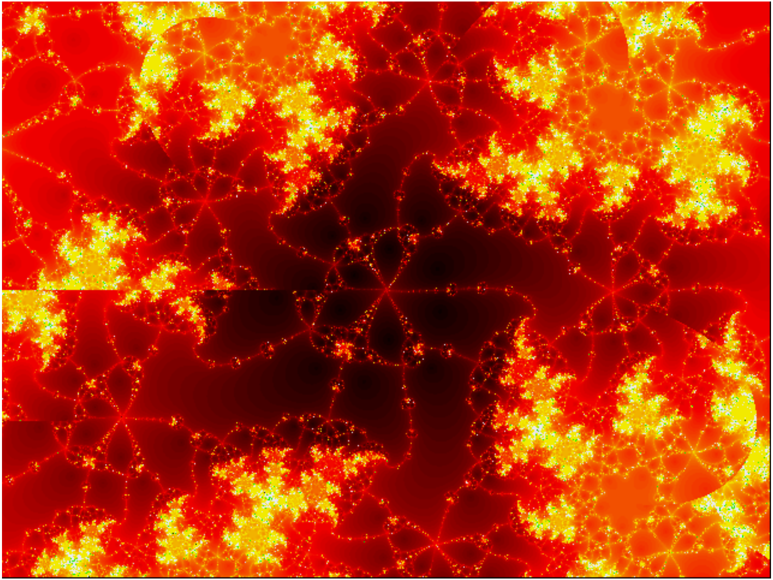
$\mathrm{Re}(z/(\cos(c \cdot \sin z)))$



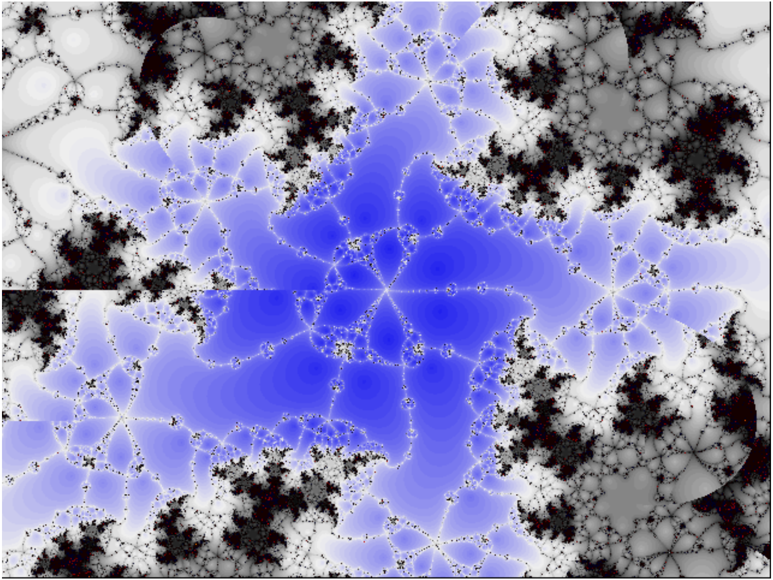
$z - (z^c + z - 1)/(cz^{c-1} + 1)$



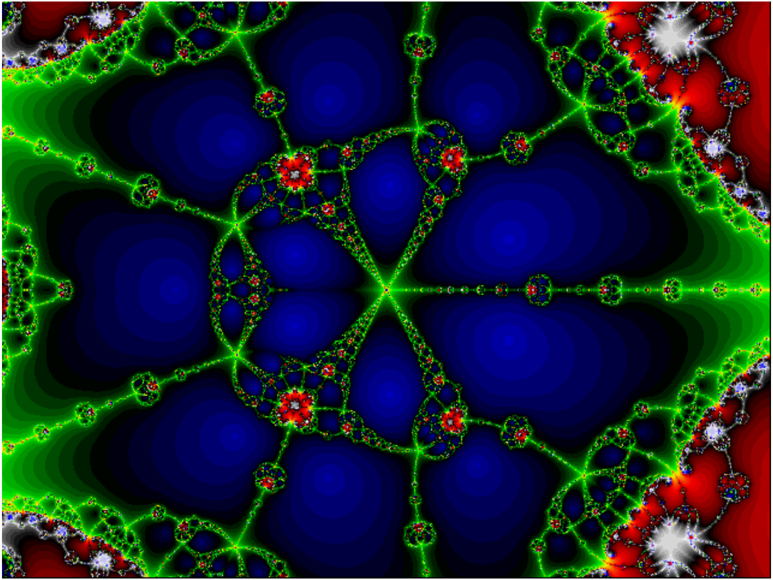
$z - (z^c + z - 1)/(cz^{c-1} + 1)$

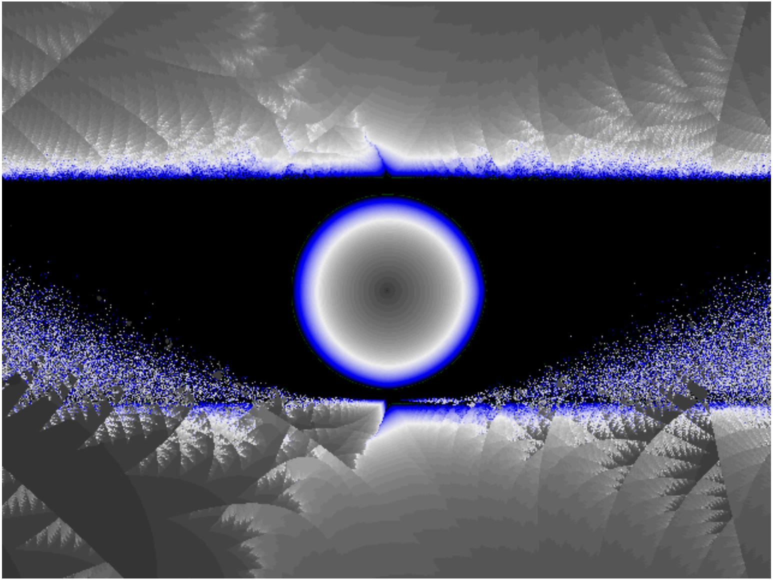


$z - (z^c + z - 1)/(cz^{c-1} + 1)$



$z - (z^c + z - 1)/(cz^{c-1} + 1)$





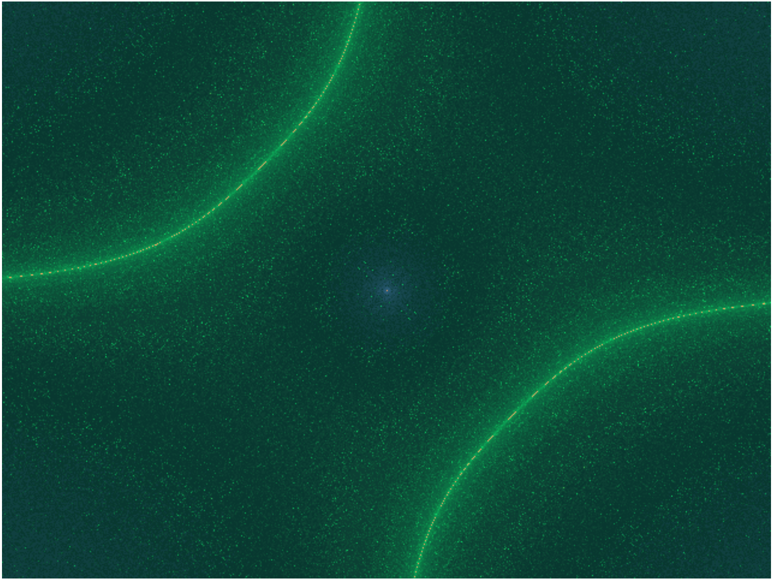
Some unusual functions

"absn" function: $\text{absn}(z) = |z| + i\,\text{Im}(z)$

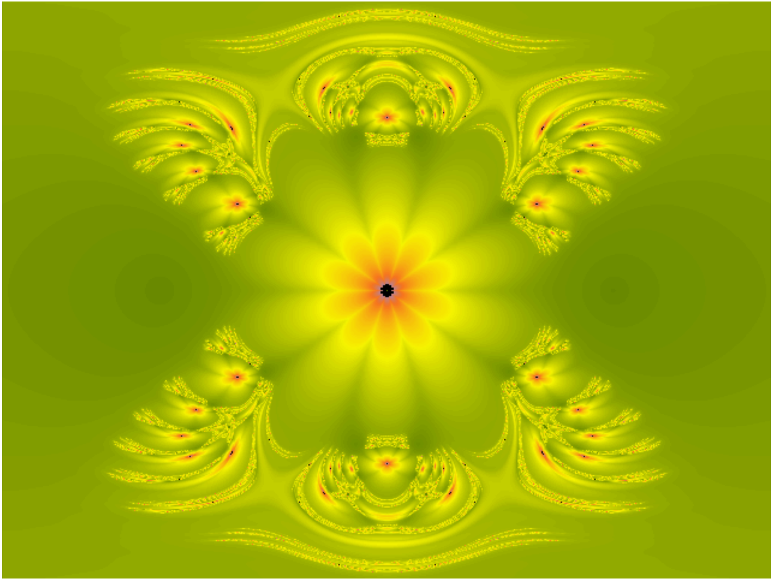
"floor" function: $\text{floor}(x + iy) = \text{floor}(x) + i \cdot \text{floor}(y)$

"and" function: $(x + iy)\&(a + ib) = (x\&a) + i(y\&b)$

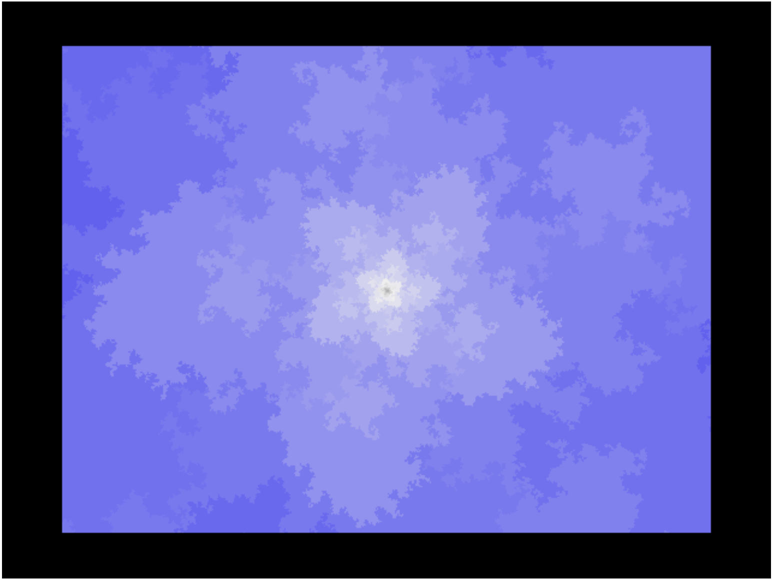
$\text{absn}(z^2) + i \cdot \text{absn}(1/z) + c$



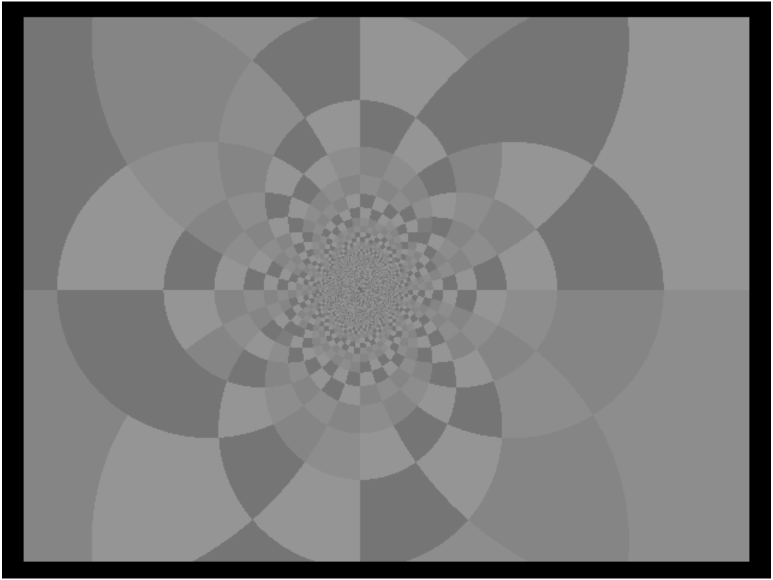
$\mathrm{absn}(z - (z^c - 1)/(cz^{c-1}))$



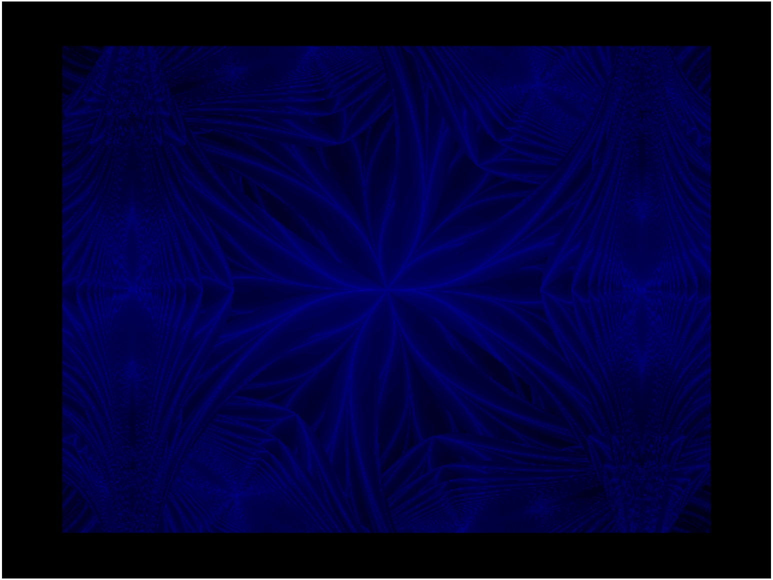
floor(cz)



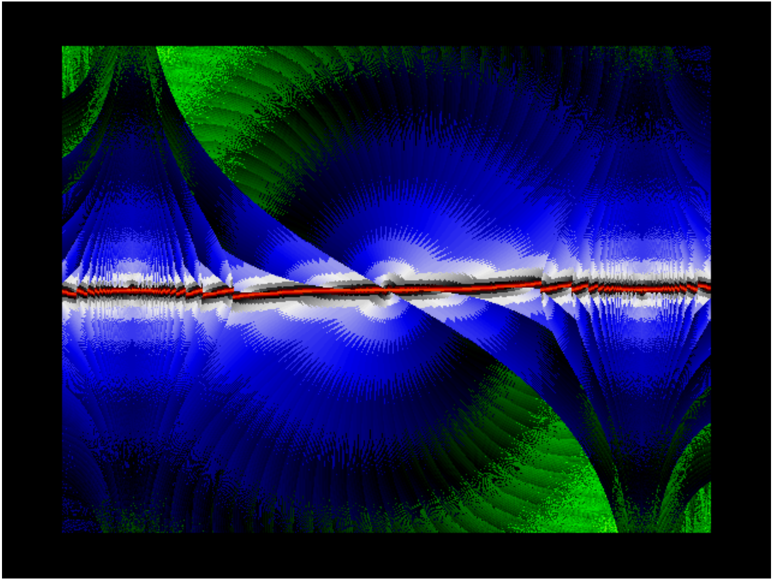
$c\,\mathrm{floor}(\sec(z))$

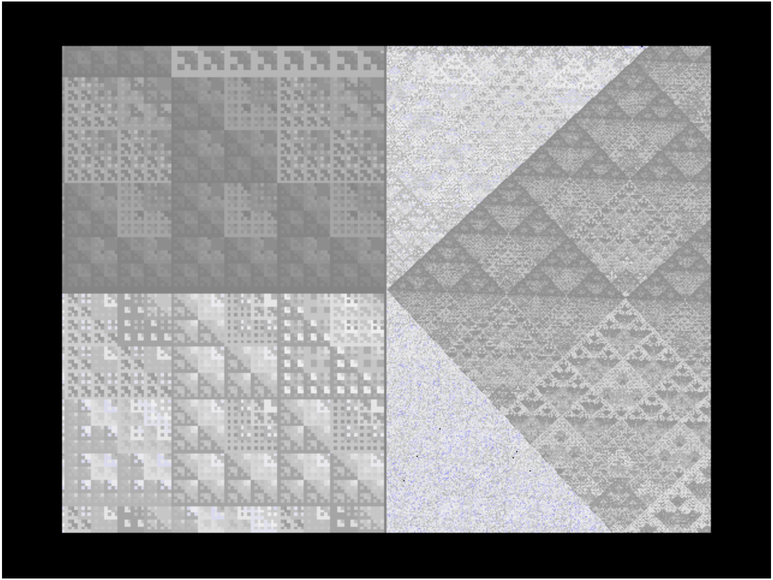


$c(x\% \operatorname{Re}(\sin(z)) + iy\% \operatorname{Im}(\sin(z)))$

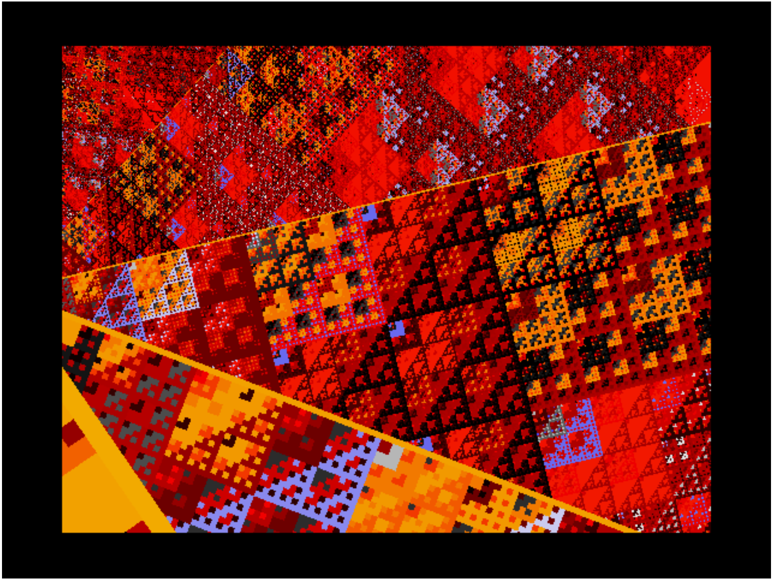


$c(x\% \operatorname{Re}(\sin(z)) + iy\% \operatorname{Im}(\sin(z)))$

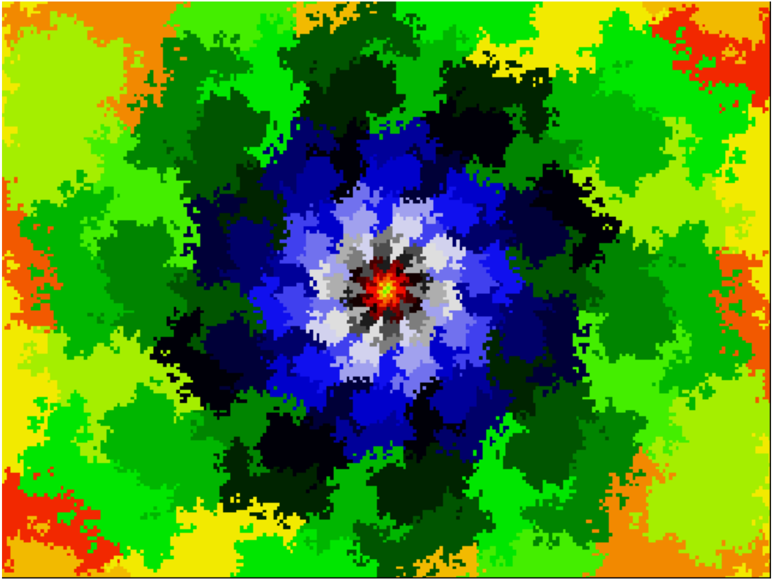




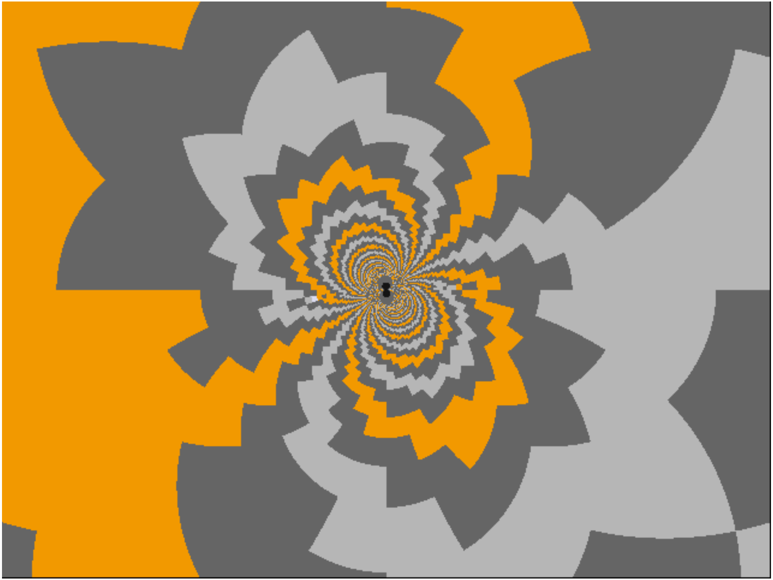
$c((x\&y)\cdot(x<0)+z\cdot(x>0))$



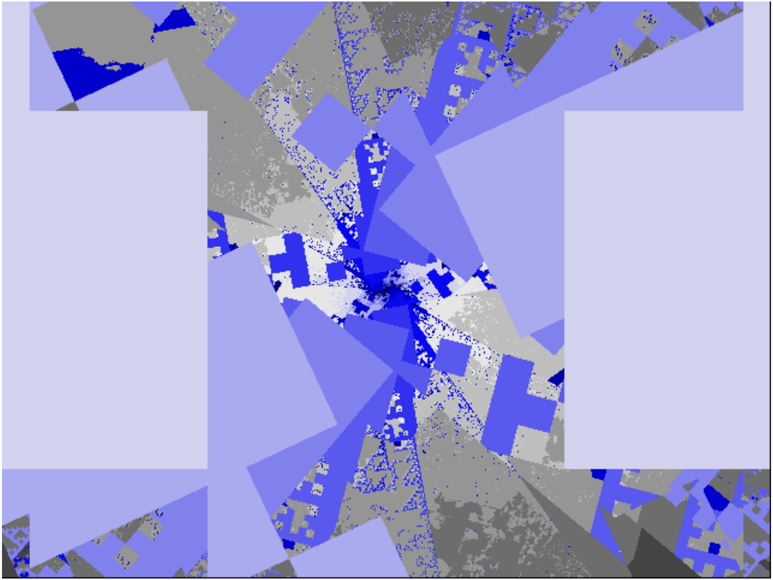
$c(\text{floor}(z) \cdot (x > 0) + \text{ceil}(z) \cdot (x < 0))$



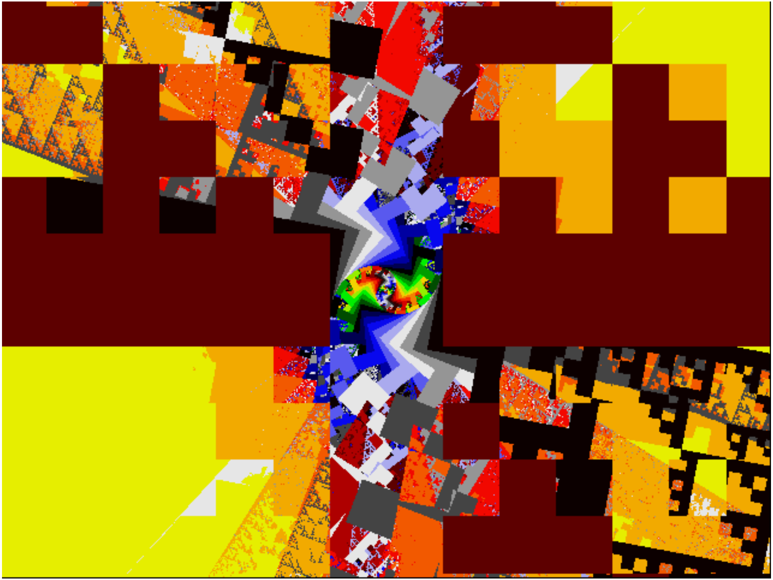
$c \cdot \text{floor}(\csc z \sec z)$



$cz(!x + (x\&y))$



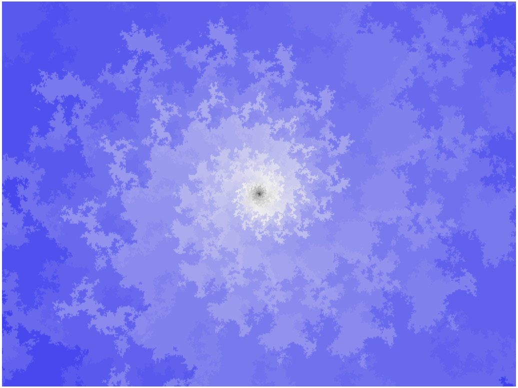
$f(x+iy) = (x+iy)(\chi_{(-1,1)}(x) + (x\&y))$, $c = .76 - .53i$



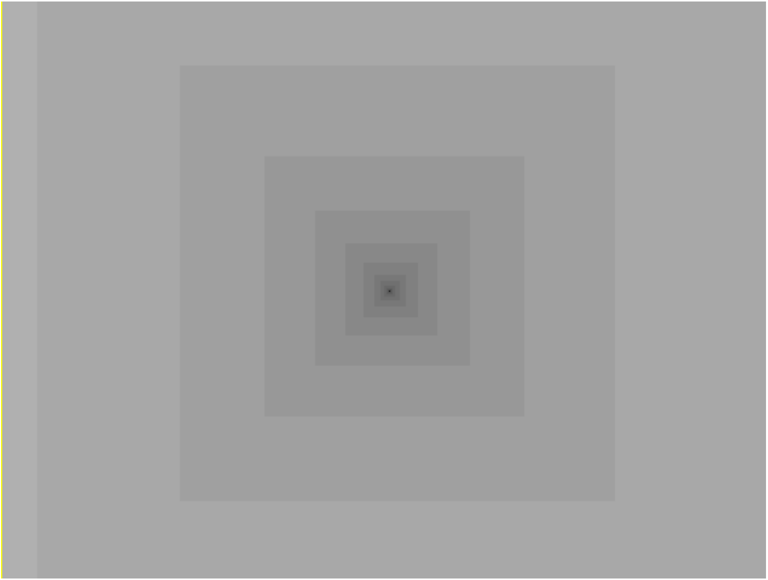
Define

$$F(z) = \lfloor x \rfloor + i\lfloor y \rfloor, \text{ where } z = x + iy.$$

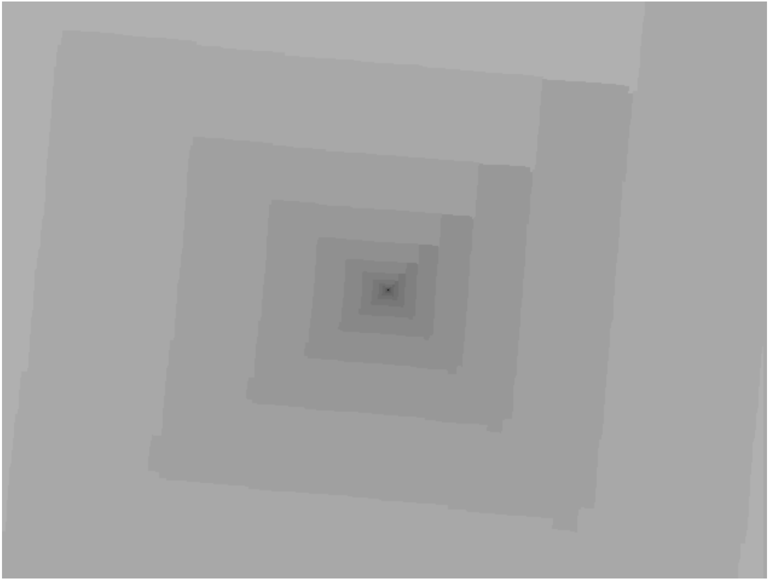
We will be iterating $cF(z)$ for various values of the constant c.



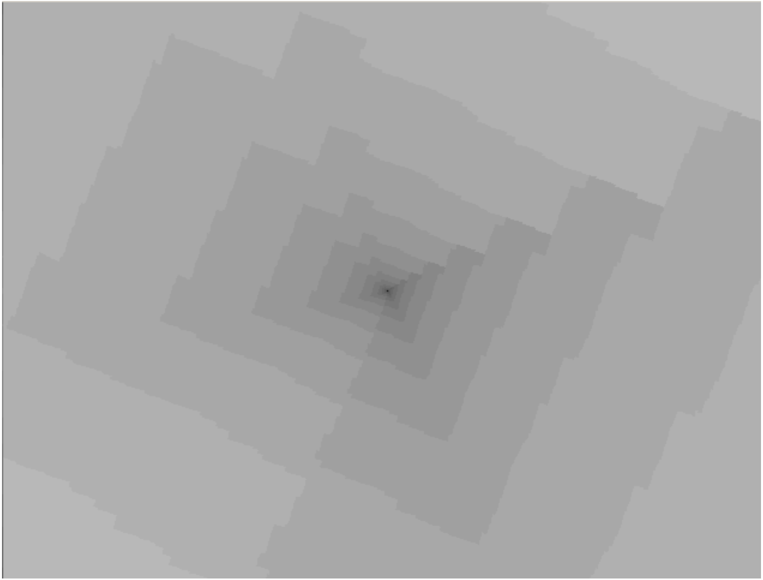
$c = .6$



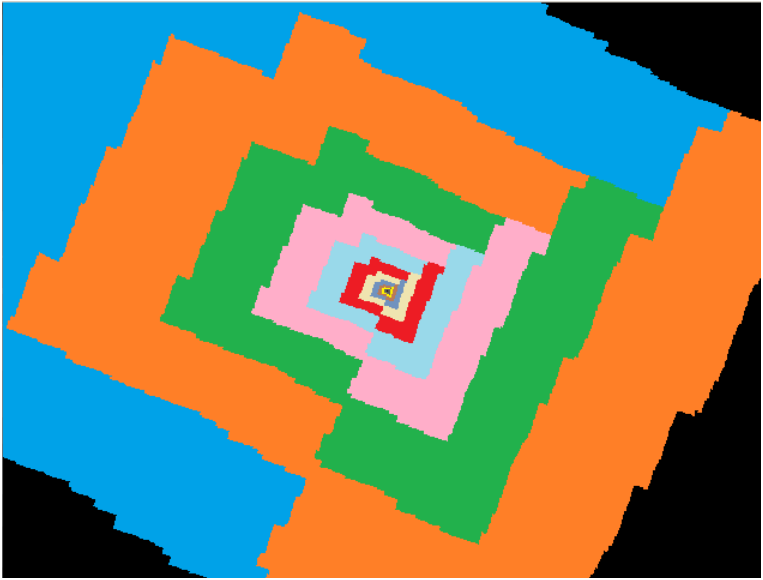
$c = .6 + .01i$



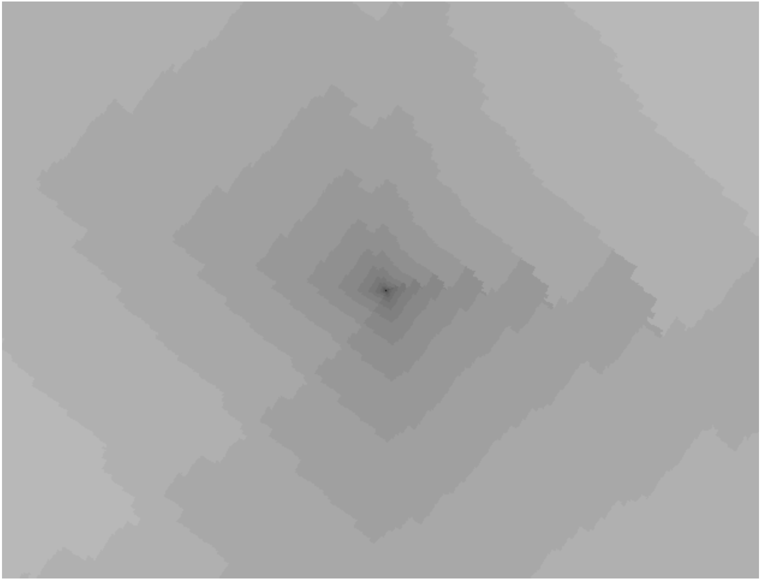
$c = .6 + .02i$



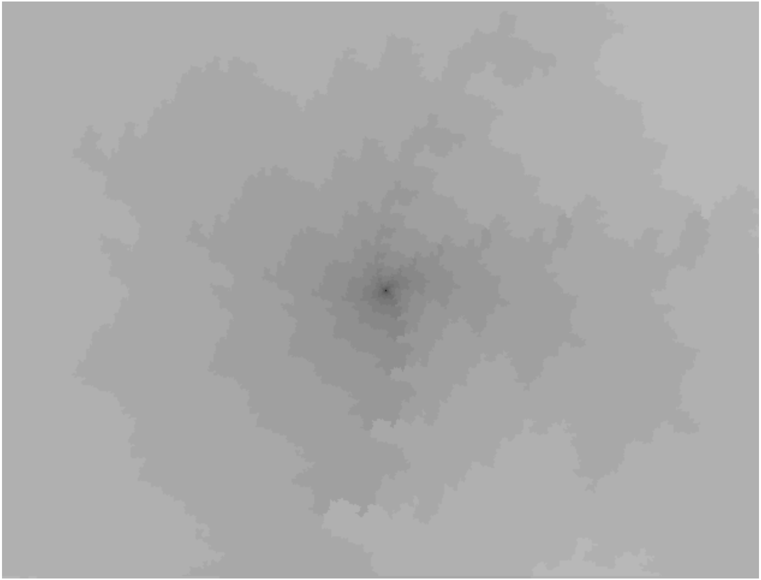
$c = .6 + .02i$ false color



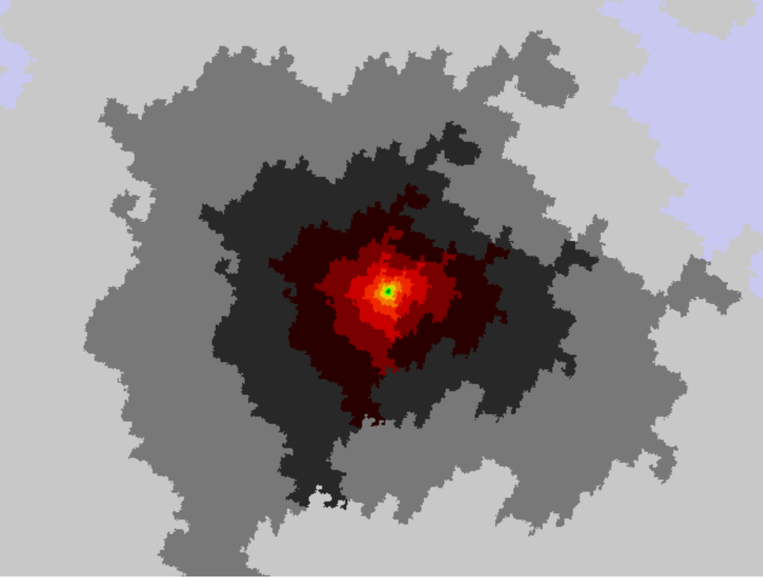
$c = .6 + .03i$



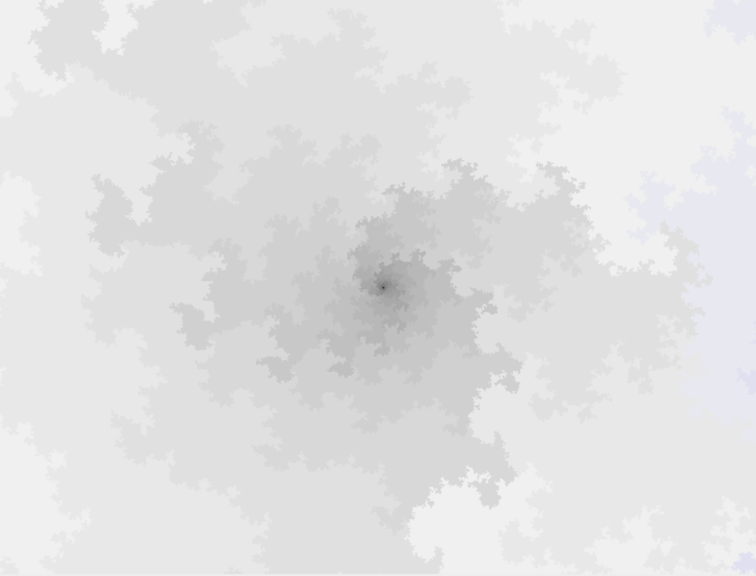
$c = .6 + .1i$



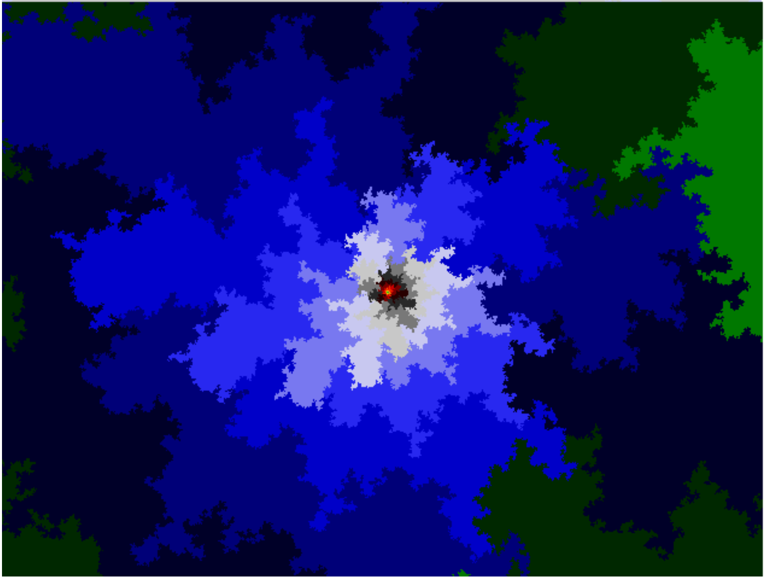
$c = .6 + .1i$ sharper gradient



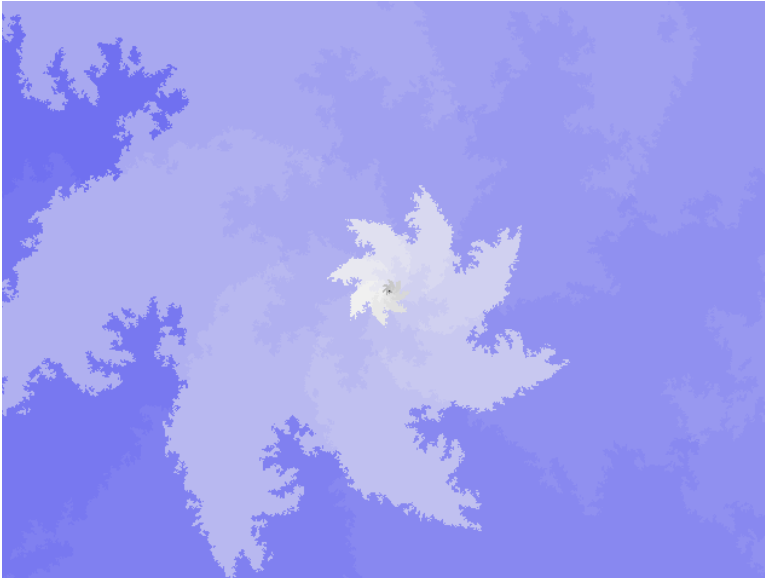
$c = .6 + .3i$



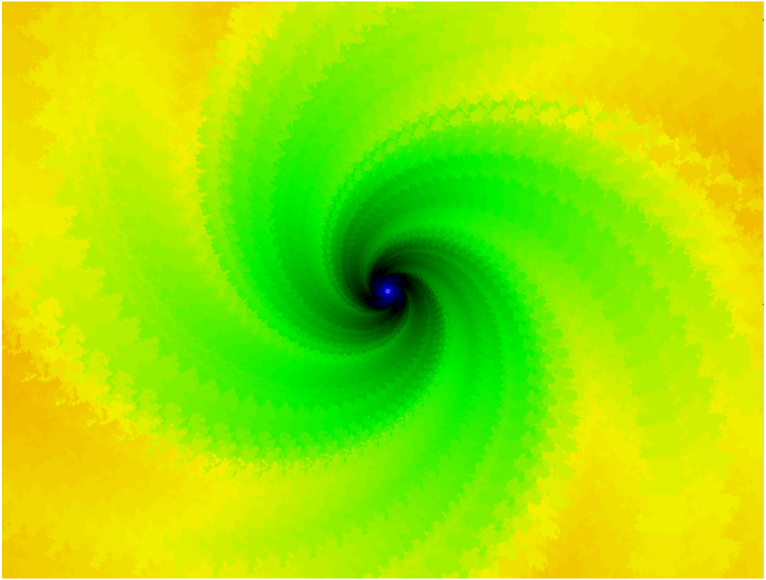
$c = .6 + .3i$ sharper gradient



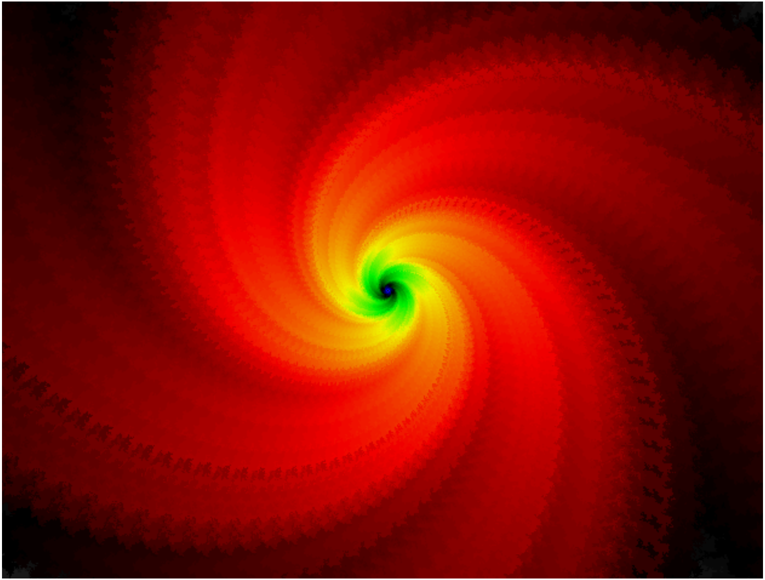
$c = .51 + .56i$



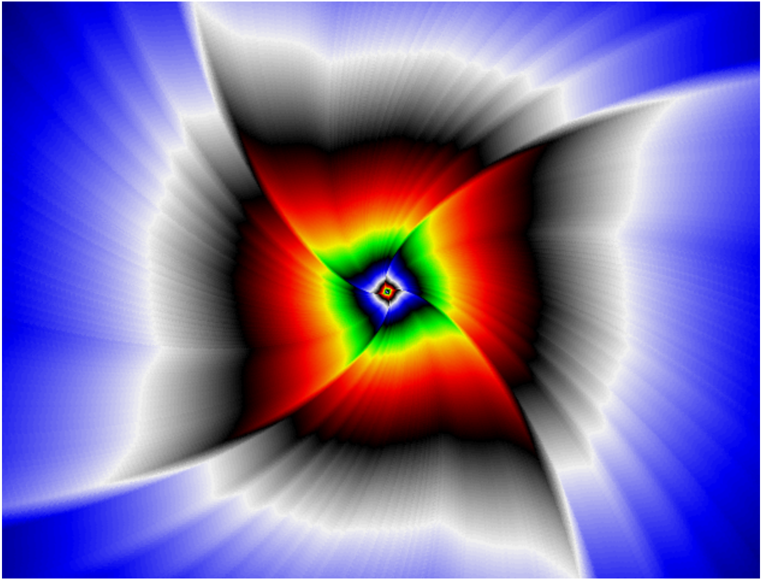
$c = .94 + .09i$



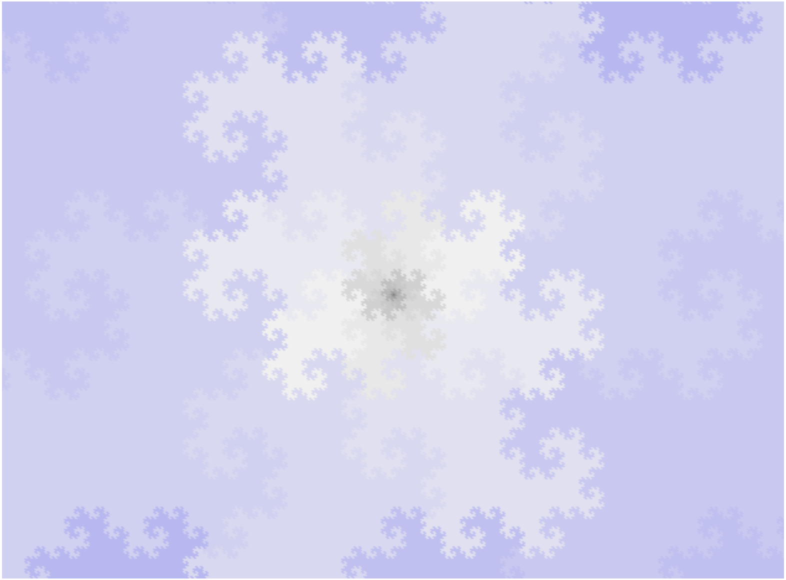
$c = .96 + .06i$



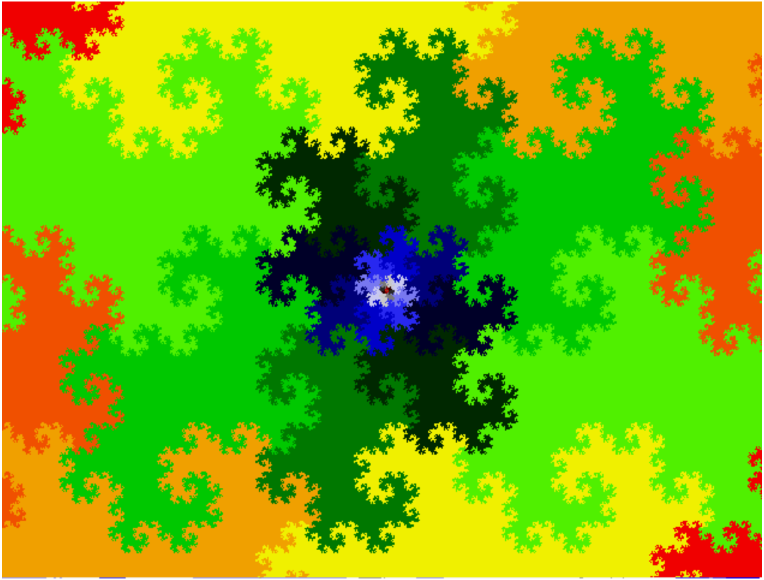
$c = .99 + .01i$



$c = .5 + .5i$

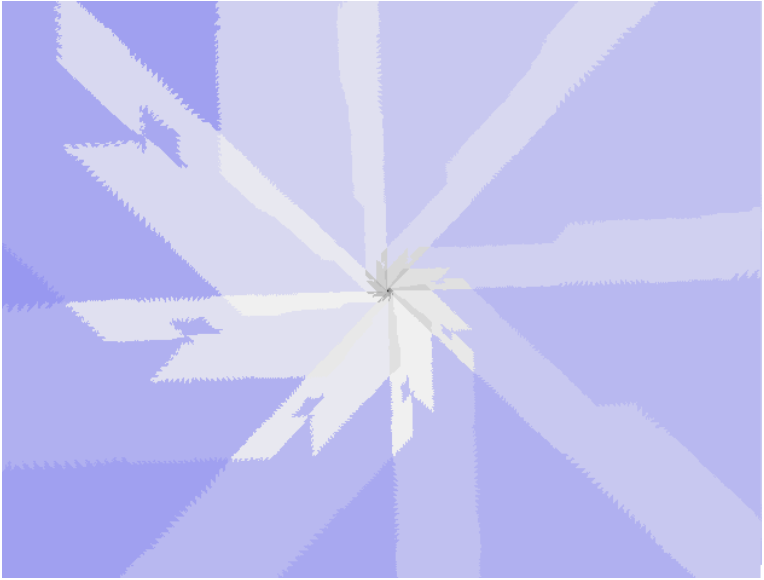


$c = .5 + .5i$

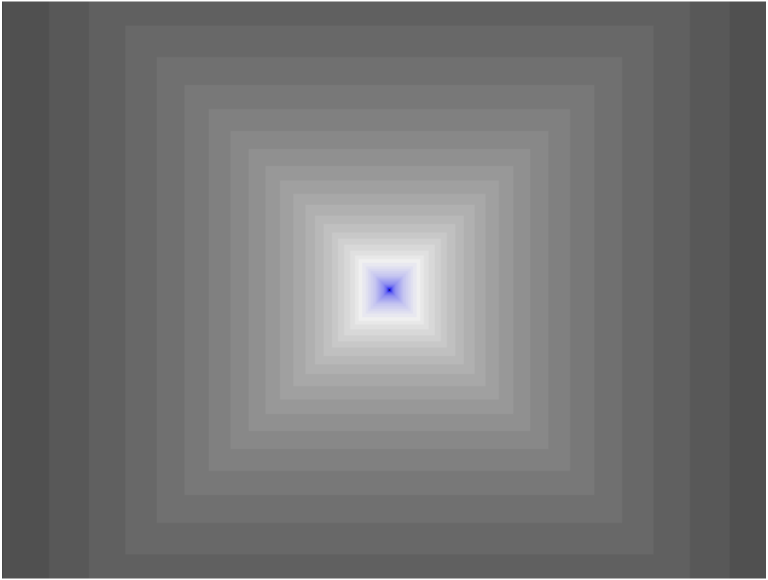


$c \approx .5 + .5i$

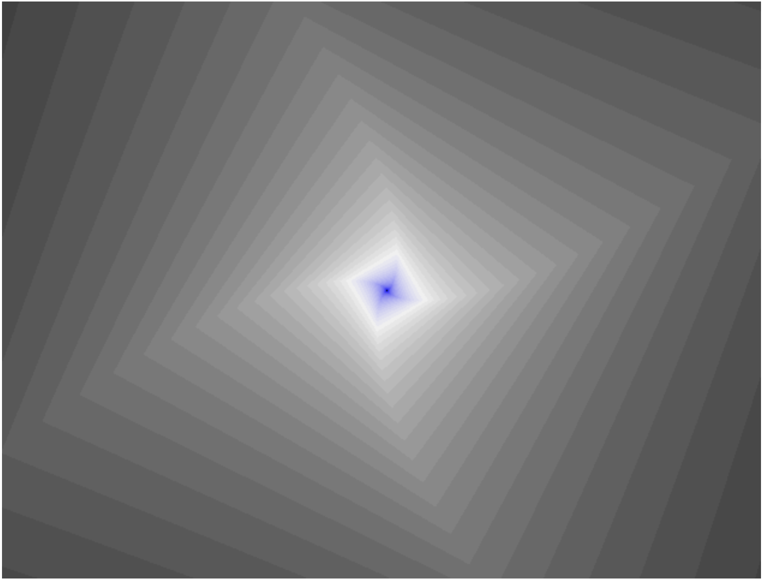
$c \approx .5 + .5i$



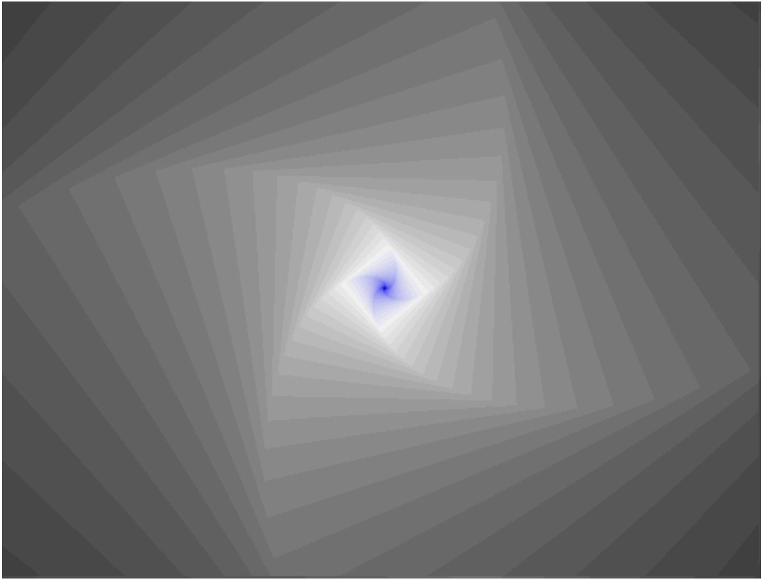
$c = 1.14$



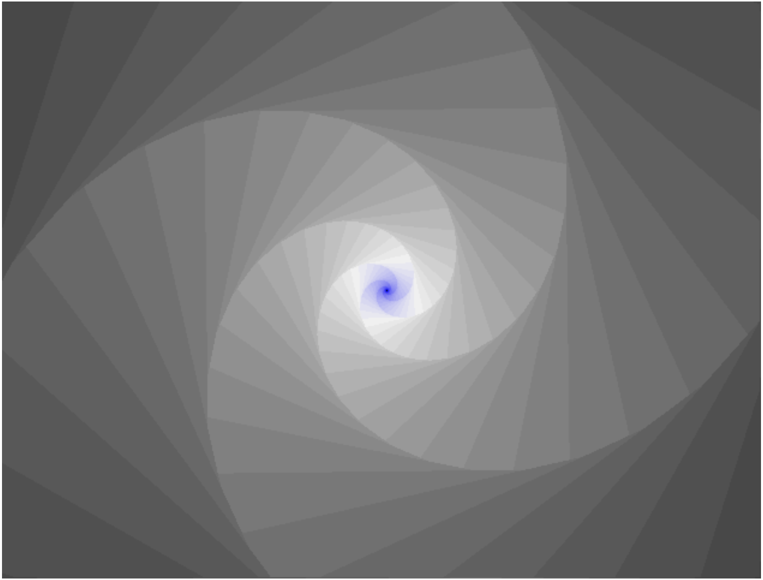
$c = 1.14 + .04i$



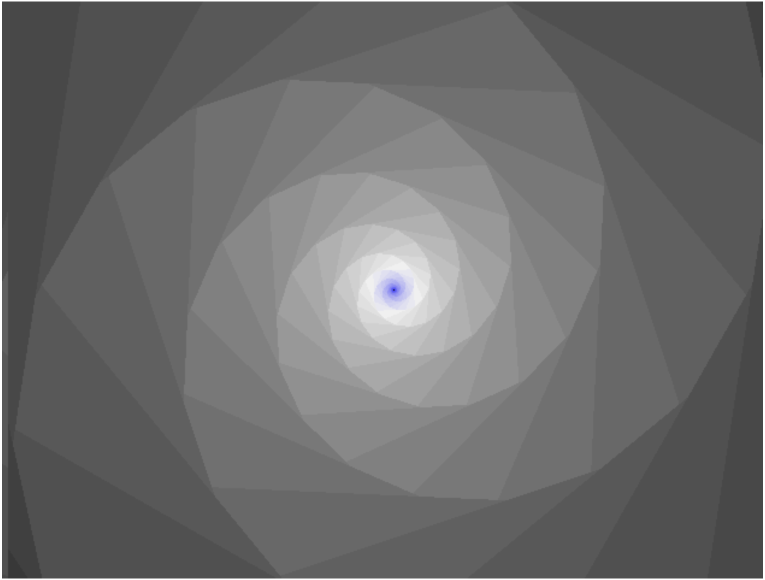
$c = 1.13 + .1i$



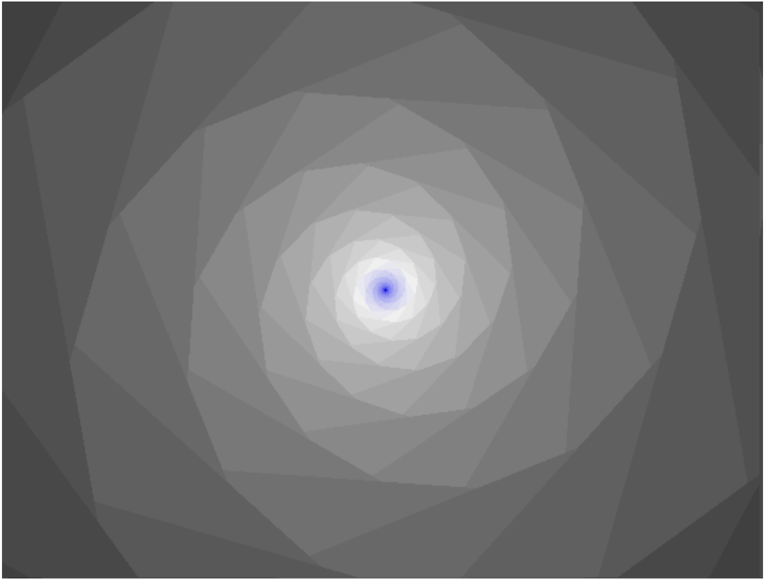
$c = 1.12 + .24i$



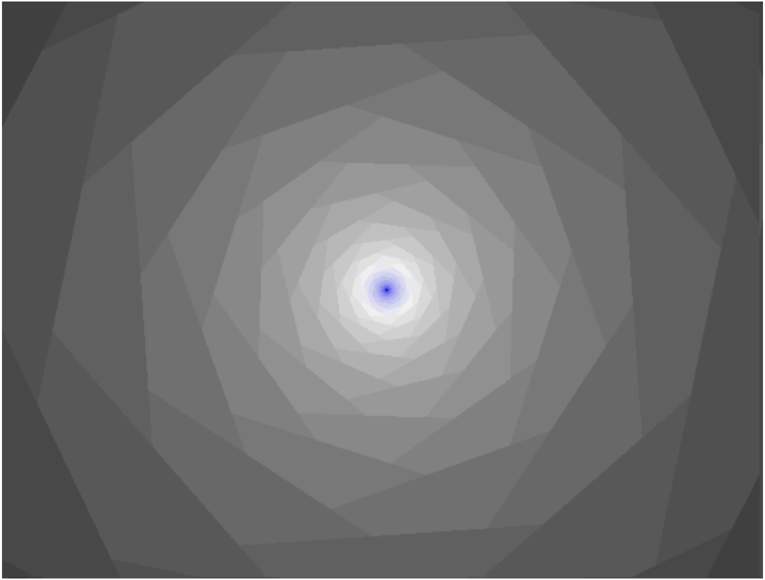
$c = 1.07 + .41i$



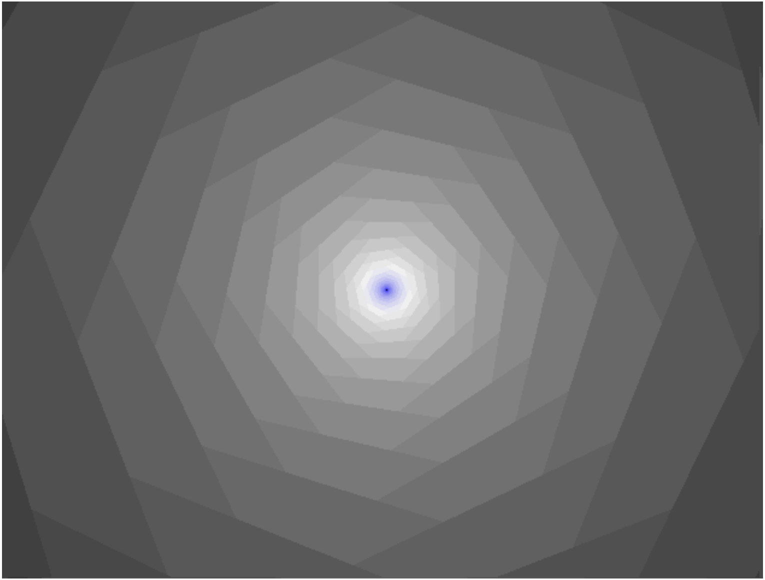
$c = 1.02 + .5i$



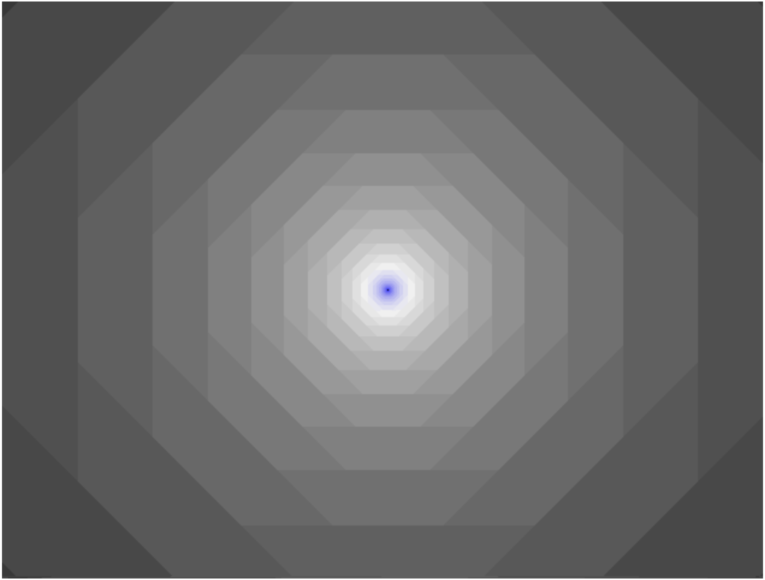
$c = .91 + .69i$



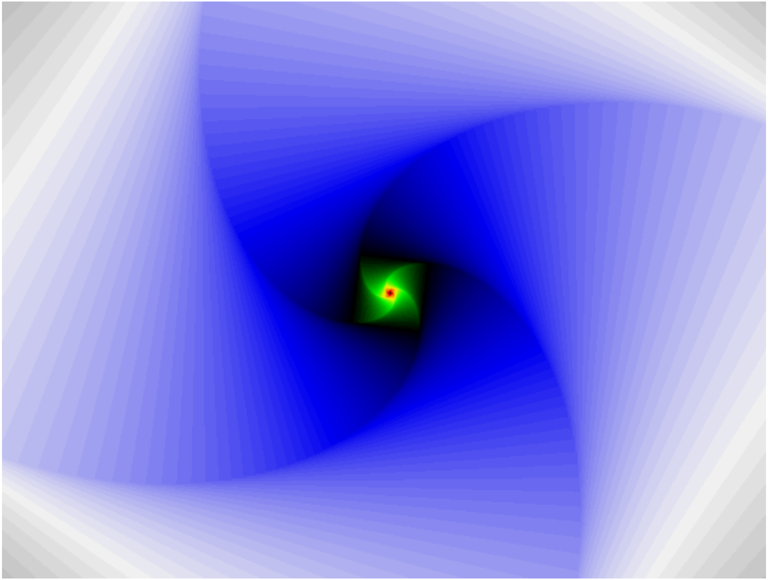
$c = .84 + .78i$



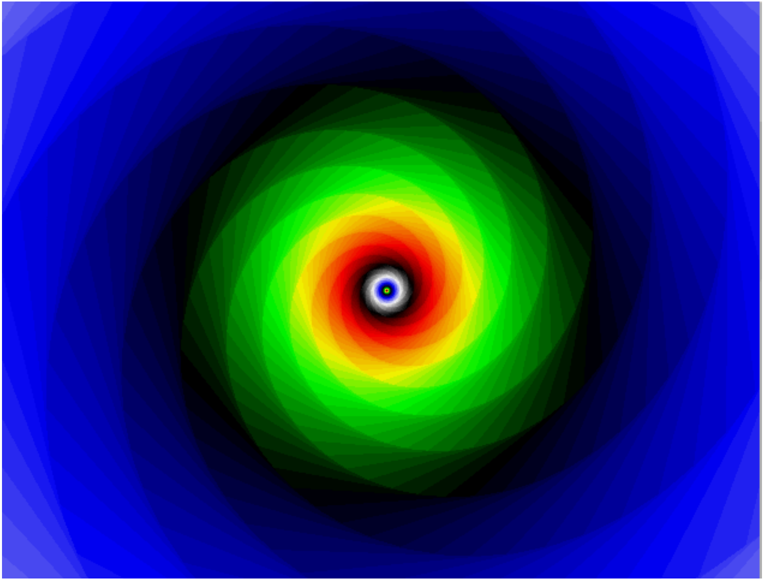
$c = .81 + .81i$



$c = .04 + 1.04i$

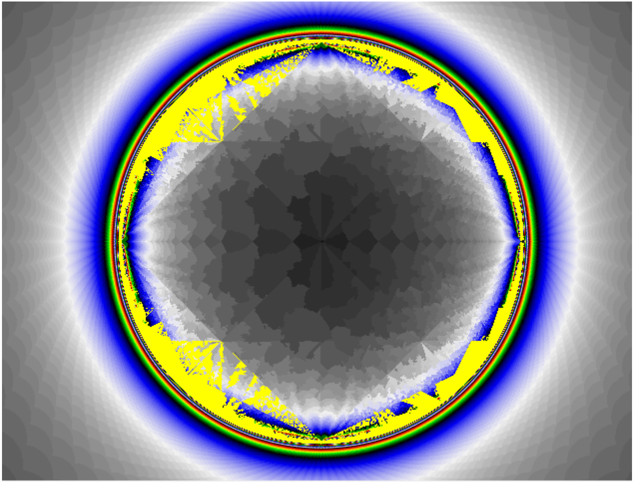


$c = .68 + .77i$

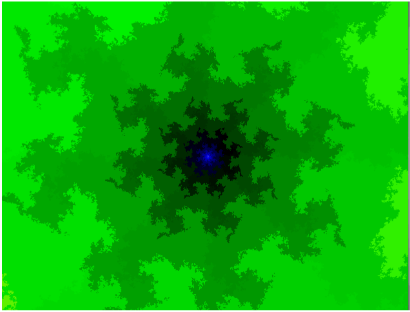


Index set

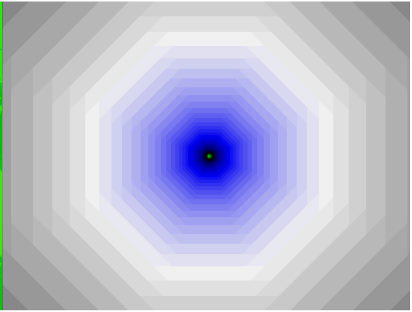
Look at what happens to the point $50 + 50i$ under iteration for various values of c.



Inside unit circle vs outside

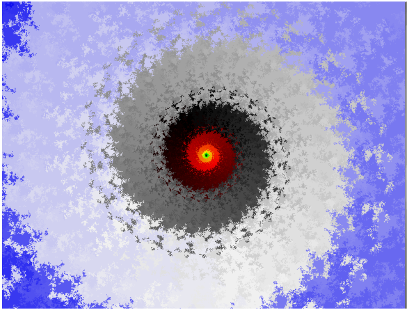


.65+.65i (inside)

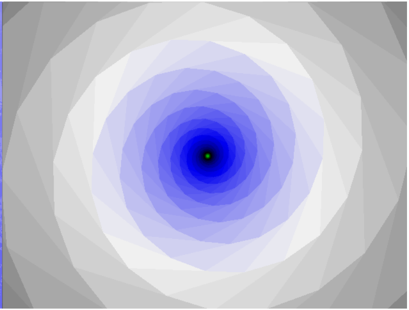


.75+.75i (outside)

Inside unit circle vs outside



.91+.31i (inside)



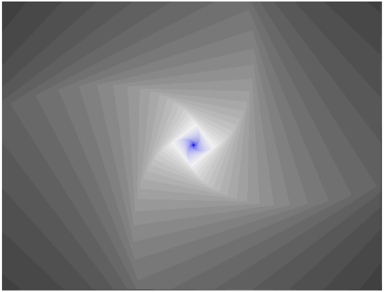
1+.34i (outside)

Outside the unit circle

Outside: Iterates attracted to ∞.

Iteration determined by relatively simple interaction between:

- Rotation from multiplying by complex values of c
- Floor function
- The norm used. Iterates "converge" to ∞ when $|x| > 10^6$ or $|y| > 10^6$. Using the Euclidean norm removes all interesting behavior.

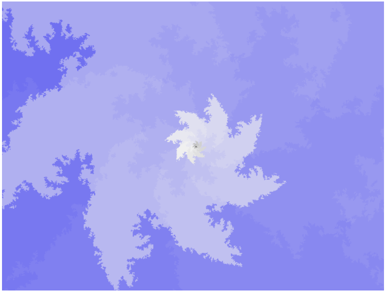


Inside the unit circle

Inside: Iterates attracted to various fixed points.

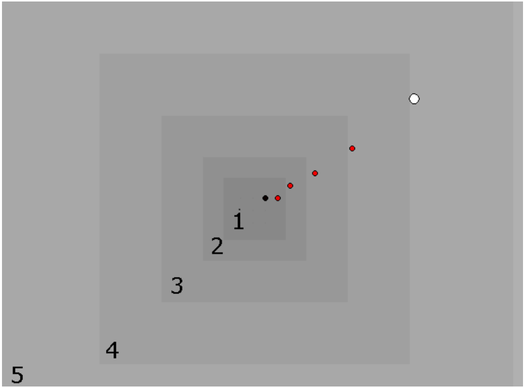
Iteration determined by

- Rotation from multiplying by complex values of c
- Floor function



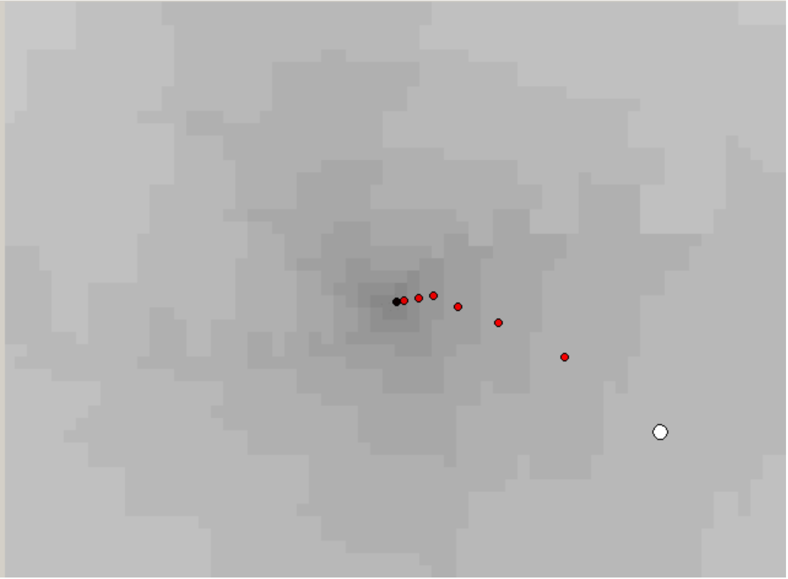
Closer look at $c = .6$

Nine fixed points: all the points of $\{-1.2, -.6, 0\} \times \{-1.2, -.6, 0\}$

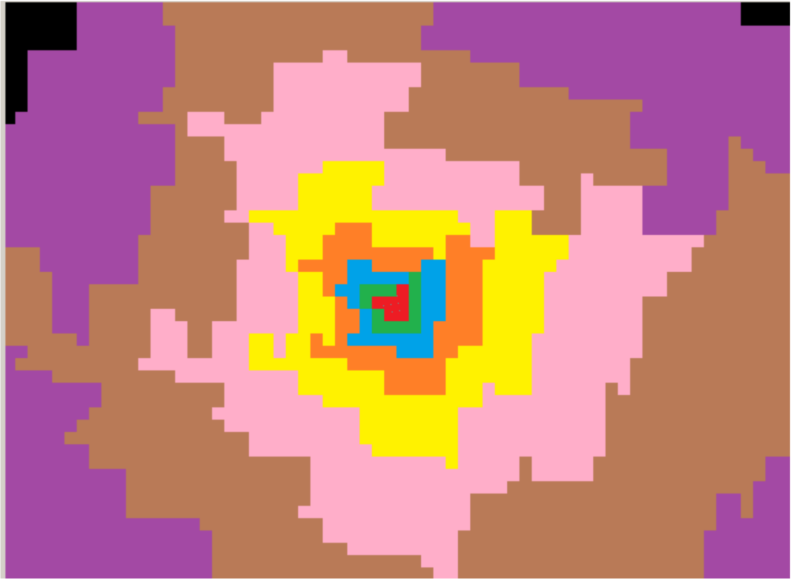


Box n = {points mapping to fixed point in n iterations}

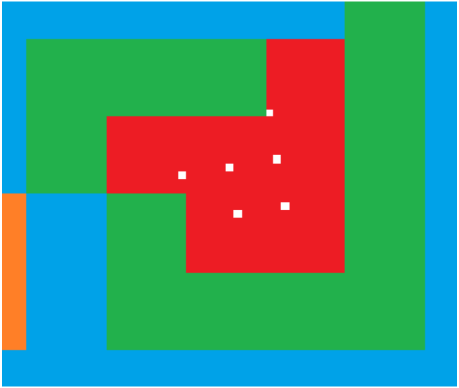
Closer look at $c = .6 + .1i$



Closer look at $c = .6 + .1i$ in false color

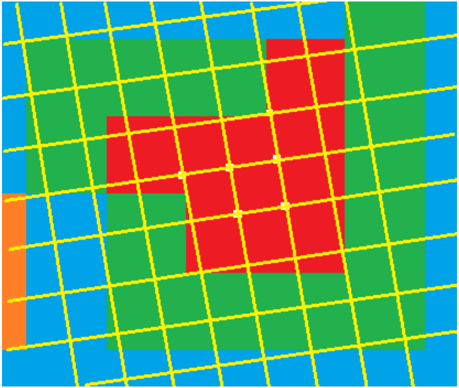


Fixed points when $c = .6 + .1i$



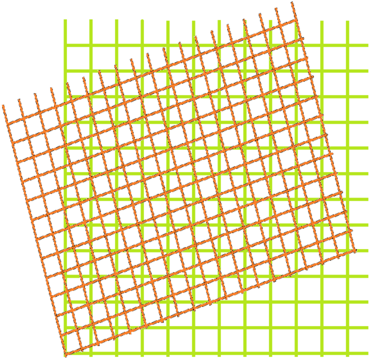
Fixed points:
$(.1, -.6), (-.5, -.7), (.2, -1.2), (0, 0), (-1.1, -.8), (-.4, -1.3)$

Slanted grid for $c = .6 + .1i$



All iterates constrained to move along slanted grid (slopes 1/6 and -6).

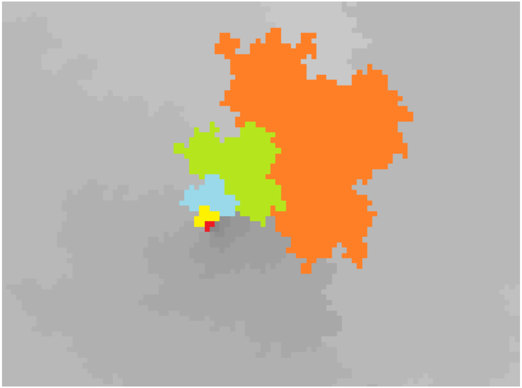
Slanted grid for $c = .6 + .1i$



Interaction between rectangular grid induced by floor and slanted grid induced by complex multiplication

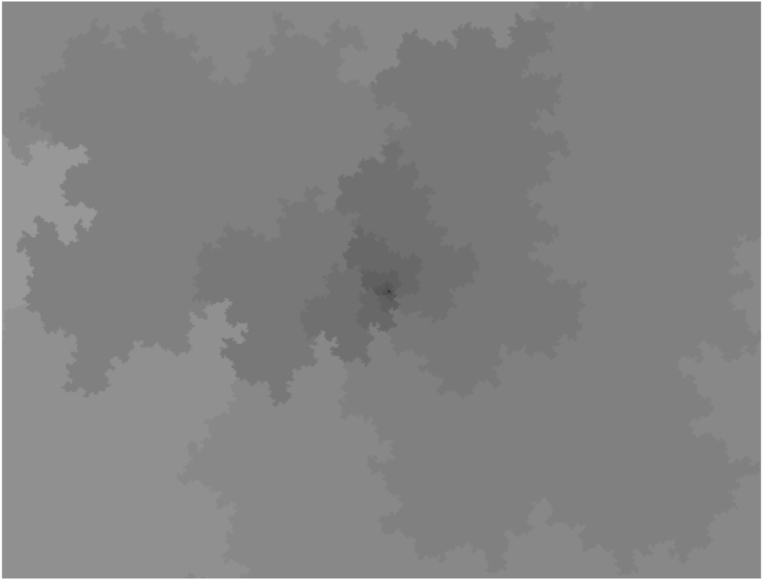
Can describe this iteration purely in terms of rotations, dilations, and "snapping to the grid."

Closer look at $c = .43 + .23i$

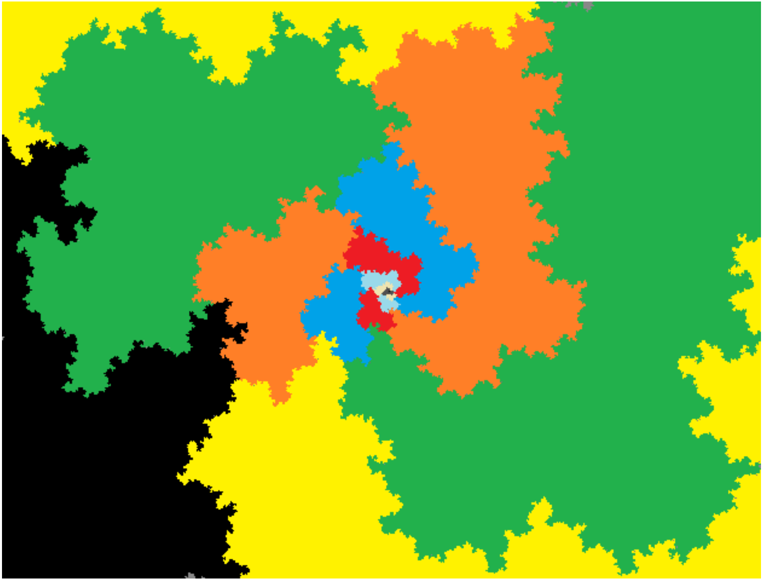


Each colored segment is a "copy" of one before it, becoming more complex in a fractal-like way.

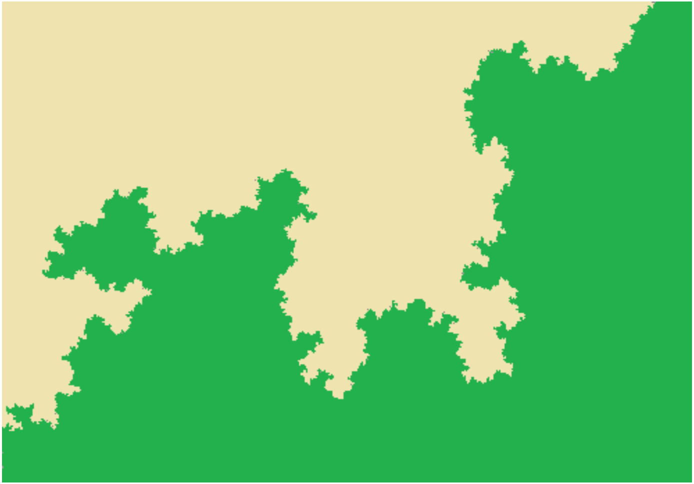
$c = .43 + .23i$



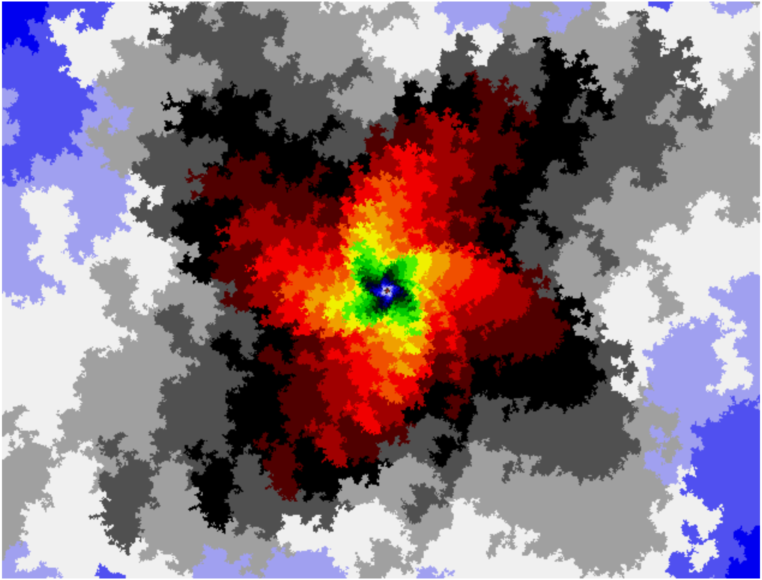
$c = .43 + .23i$ false color



Far zoom out of a section from $c = .43 + .23i$



$c = .78 + .14i$ sharper gradient



$c = .64 + .34i$ sharper gradient